Well-quasi-orders in Logic

Sylvain Schmitz

Logic Mentoring Workshop, June 22, 2019

Outline

well-quasi-orders (wqo):

- robust notion
- selection of applications:
  - algorithm termination
  - relevance logic
  - preservation theorems
  - certain answers

research projects

Outline

well-quasi-orders (wqo):

- robust notion
- selection of applications:
  - algorithm termination
  - relevance logic
  - preservation theorems
  - certain answers

research projects

Outline

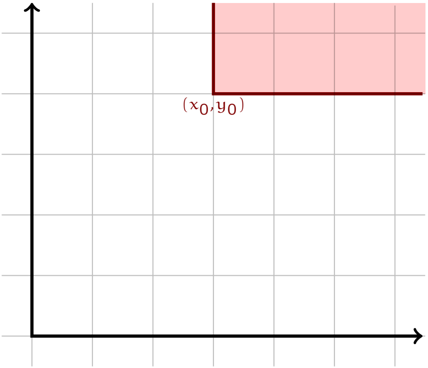
well-quasi-orders (wqo):

- robust notion
- selection of applications:
  - algorithm termination
  - relevance logic
  - preservation theorems
  - certain answers

research projects

A One-Player Game

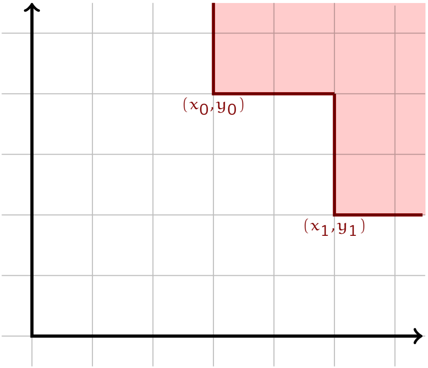
- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?

A One-Player Game

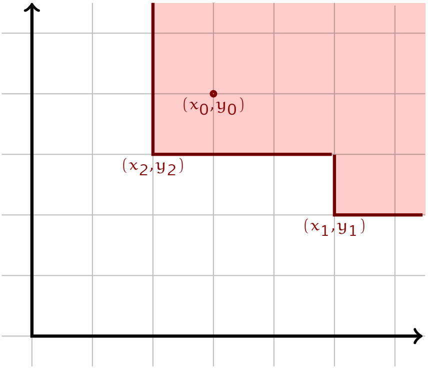
- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?

A One-Player Game

- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$

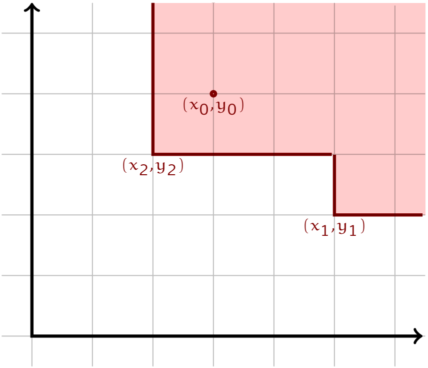


- Can Eloise win, i.e. play indefinitely?

If $(x_0, y_0) \neq (0, 0)$, then choosing $(x_j, y_j) = (\frac{x_0}{2^j}, \frac{y_0}{2^j})$ wins.

A One-Player Game

- over $\mathbb{N} \times \mathbb{N}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?

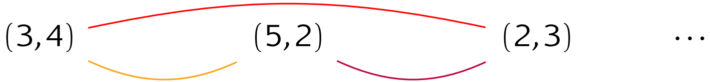
Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$.

Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$. Consider the pairs of indices $i < j$: color (i, j)

purple if $x_i > x_j$ but $y_i \leqslant y_j$,

red if $x_i > x_j$ and $y_i > y_j$,

orange if $y_i > y_j$ but $x_i \leqslant x_j$.

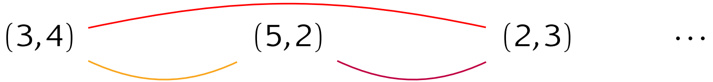


Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$. Consider the pairs of indices $i < j$: color (i, j)

purple if $x_i > x_j$ but $y_i \leqslant y_j$,

red if $x_i > x_j$ and $y_i > y_j$,

orange if $y_i > y_j$ but $x_i \leqslant x_j$.



By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices.

Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$. Consider the pairs of indices $i < j$: color (i,j)

purple if $x_i > x_j$ but $y_i \leqslant y_j$,

red if $x_i > x_j$ and $y_i > y_j$,

orange if $y_i > y_j$ but $x_i \leqslant x_j$.

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices. In all cases, it implies the existence of an infinite decreasing sequence in $\mathbb{N}$, a contradiction.

Well-Quasi-Orders

- multiple equivalent definitions
- algebraic constructions

Well-Quasi-Orders

- multiple equivalent definitions: $(X, \leqslant)$ wqo iff
  - bad sequences are finite: $x_0, x_1, \ldots$ is bad if $\forall i < j, x_i \not\leqslant x_j$
  - $\leqslant$ is well-founded and has no infinite antichains
  - finite basis property: $\emptyset \subsetneq U \subseteq X$ has at least one and finitely many minimal elements
  - ascending chain condition: any chain $U_0 \subsetneq U_1 \subsetneq \cdots$ of upwards-closed sets is finite
  - etc.
- algebraic constructions

Well-Quasi-Orders

- multiple equivalent definitions: $(X, \leqslant)$ wqo iff
  - bad sequences are finite: $x_0, x_1, \ldots$ is bad if $\forall i < j, x_i \not\leqslant x_j$
  - $\leqslant$ is well-founded and has no infinite antichains
  - finite basis property: $\emptyset \subsetneq U \subseteq X$ has at least one and finitely many minimal elements
  - ascending chain condition: any chain $U_0 \subsetneq U_1 \subsetneq \cdots$ of upwards-closed sets is finite
  - etc.
- algebraic constructions

Well-Quasi-Orders

- multiple equivalent definitions: $(X, \leqslant)$ wqo iff
  - bad sequences are finite: $x_0, x_1, \ldots$ is bad if $\forall i < j, x_i \not\leqslant x_j$
  - $\leqslant$ is well-founded and has no infinite antichains
  - finite basis property: $\emptyset \subsetneq U \subseteq X$ has at least one and finitely many minimal elements
  - ascending chain condition: any chain $U_0 \subsetneq U_1 \subsetneq \cdots$ of upwards-closed sets is finite
  - etc.
- algebraic constructions

Well-Quasi-Orders

- multiple equivalent definitions: $(X, \leqslant)$ wqo iff
  - bad sequences are finite: $x_0, x_1, \ldots$ is bad if $\forall i < j, x_i \not\leqslant x_j$
  - $\leqslant$ is well-founded and has no infinite antichains
  - finite basis property: $\emptyset \subsetneq U \subseteq X$ has at least one and finitely many minimal elements
  - ascending chain condition: any chain $U_0 \subsetneq U_1 \subsetneq \cdots$ of upwards-closed sets is finite
  - etc.
- algebraic constructions

Well-Quasi-Orders

- multiple equivalent definitions: $(X, \leqslant)$ wqo iff
  - bad sequences are finite: $x_0, x_1, \ldots$ is bad if $\forall i < j, x_i \not\leqslant x_j$
  - $\leqslant$ is well-founded and has no infinite antichains
  - finite basis property: $\emptyset \subsetneq U \subseteq X$ has at least one and finitely many minimal elements
  - ascending chain condition: any chain $U_0 \subsetneq U_1 \subsetneq \cdots$ of upwards-closed sets is finite
  - etc.
- algebraic constructions

Well-Quasi-Orders

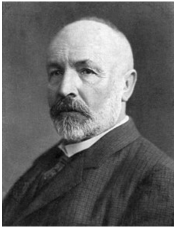
- multiple equivalent definitions
- algebraic constructions
  - Cartesian products (Dickson's Lemma),
  - finite sequences (Higman's Lemma),
  - disjoint sums,
  - finite sets with Hoare's quasi-ordering,
  - finite trees (Kruskal's Tree Theorem),
  - graphs with minors (Robertson and Seymour's Graph Minor Theorem),
  - etc.

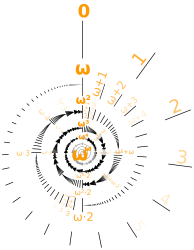
Example: Ordinals

ordinal: well-founded linear order

bad sequences are descending sequences:

$$\alpha \not\leq \beta \text{ iff } \alpha > \beta$$





Example: Dickson's Lemma

Lemma (Dickson 1913)

If $(X, \leqslant_X)$ *and* $(Y, \leqslant_Y)$ *are two wqos, then* $(X \times Y, \leqslant_\times)$ *is a wqo, where* $\leqslant_\times$ *is the product ordering:*

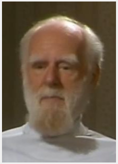
$$\langle x, y\rangle \leqslant_\times \langle x', y'\rangle \stackrel{\text{def}}{\Leftrightarrow} x \leqslant_X x' \wedge y \leqslant_Y y' .$$

Example

- $(\mathbb{N}^d, \leqslant_\times)$ using the product ordering
- $(\mathbb{M}(X), \leqslant_m)$ for finite multiset embedding over finite X

Example: Higman's Lemma

Lemma (Higman 1952)



If $(X, \leqslant)$ is a wqo, then $(X^*, \leqslant_*)$ is a wqo where $\leqslant_*$ is the subword embedding ordering:

$$a_1 \cdots a_m \leqslant_* b_1 \cdots b_n \overset{\text{def}}{\Leftrightarrow} \begin{cases} \exists 1 \leqslant i_1 < \cdots < i_m \leqslant n, \\ \bigwedge_{j=1}^{m} a_j \leqslant_A b_{i_j} \ . \end{cases}$$

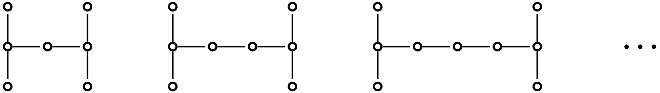
Example

$$aba \leqslant_* baaacabbab$$

Example: Bounded Tree-Depth

Lemma (Ding 1992)
For all k, $(\text{Graphs} \setminus \uparrow P_k, \subseteq)$ *is wqo.*

Non-Examples



Application: Algorithm Termination

```
SIMPLE (a,b)
c ⟵ 1
while a > 0 ∧ b > 0
    ⟨a,b,c⟩ ⟵ ⟨a−1,b,2c⟩
  or
    ⟨a,b,c⟩ ⟵ ⟨2c,b−1,1⟩
end
```

$\langle a_0,b_0,c_0\rangle$

$\langle a_1,b_1,c_1\rangle$

$\vdots$

$\langle a_i,b_i,c_i\rangle$

$\vdots$

$\langle a_j,b_j,c_j\rangle$

- in any execution, $\langle a_0,b_0\rangle,\ldots,\langle a_n,b_n\rangle$ is a bad sequence over $(\mathbb{N}^2,\leqslant_\times)$,
- $(\mathbb{N}^2,\leqslant_\times)$ is a wqo: all the runs are finite
- c.f. Podelski & Rybalchenko's transition invariants

Application: Algorithm Termination

```
SIMPLE (a, b)
c ⟵ 1
while a > 0 ∧ b > 0
    ⟨a, b, c⟩ ⟵ ⟨a − 1, b, 2c⟩
  or
    ⟨a, b, c⟩ ⟵ ⟨2c, b − 1, 1⟩
end
```

$\langle a_0, b_0, c_0 \rangle$

$\langle a_1, b_1, c_1 \rangle$

$\vdots$

$\langle a_i, b_i, c_i \rangle$

$\vdots \quad \not\leqslant_\times$

$\langle a_j, b_j, c_j \rangle$

- in any execution, $\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle$ is a bad sequence over $(\mathbb{N}^2, \leqslant_\times)$,
- $(\mathbb{N}^2, \leqslant_\times)$ is a wqo: all the runs are finite
- c.f. Podelski & Rybalchenko's transition invariants

Application: Algorithm Termination

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
  $\langle a, b, c \rangle \longleftarrow \langle a-1, b, 2c \rangle$
 or
  $\langle a, b, c \rangle \longleftarrow \langle 2c, b-1, 1 \rangle$
end

$\langle a_0, b_0, c_0 \rangle$
$\langle a_1, b_1, c_1 \rangle$
$\vdots$
$\langle a_i, b_i, c_i \rangle$
$\vdots$ $\not\leqslant_\times$
$\langle a_j, b_j, c_j \rangle$

- in any execution, $\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle$ is a bad sequence over $(\mathbb{N}^2, \leqslant_\times)$,
- $(\mathbb{N}^2, \leqslant_\times)$ is a wqo: all the runs are finite
- c.f. Podelski & Rybalchenko's transition invariants

Application: Algorithm Termination

SIMPLE (a, b)

$c \longleftarrow 1$

while $a > 0 \wedge b > 0$

$\quad \langle a, b, c \rangle \longleftarrow \langle a - 1, b, 2c \rangle$

or

$\quad \langle a, b, c \rangle \longleftarrow \langle 2c, b - 1, 1 \rangle$

end

$$\langle a_0, b_0, c_0 \rangle$$
$$\langle a_1, b_1, c_1 \rangle$$
$$\vdots$$
$$\langle a_i, b_i, c_i \rangle$$
$$\vdots \quad \not\leqslant_\times$$
$$\langle a_j, b_j, c_j \rangle$$

- in any execution, $\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle$ is a bad sequence over $(\mathbb{N}^2, \leqslant_\times)$,
- $(\mathbb{N}^2, \leqslant_\times)$ is a wqo: all the runs are finite
- c.f. Podelski & Rybalchenko's transition invariants

Application: Algorithm Termination

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
  $\langle a, b, c \rangle \longleftarrow \langle a - 1, b, 2c \rangle$
 or
  $\langle a, b, c \rangle \longleftarrow \langle 2c, b - 1, 1 \rangle$
end

$\langle a_0, b_0, c_0 \rangle$
$\langle a_1, b_1, c_1 \rangle$
$\vdots$
$\langle a_i, b_i, c_i \rangle$
$\vdots$ $\not\leqslant_\times$
$\langle a_j, b_j, c_j \rangle$

- in any execution, $\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle$ is a bad sequence over $(\mathbb{N}^2, \leqslant_\times)$,
- $(\mathbb{N}^2, \leqslant_\times)$ is a wqo: all the runs are finite
- c.f. Podelski & Rybalchenko's transition invariants


Application: Relevance Logic

Example $(A \to (B \to A))$

"if it's raining (A), then if your favorite color is green (B) then it's raining (A)"

A theorem in classical logic, not in relevance logic.

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\,(\mathrm{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,(\mathrm{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\,(\to_L) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to_R)$$

Application: Relevance Logic

Example $(A \to (B \to A))$

"if it's raining (A), then if your favorite color is green (B) then it's raining (A)"

A theorem in classical logic, not in relevance logic.

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\ (\mathsf{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (\mathsf{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\ (\to_{\mathsf{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to_{\mathsf{R}})$$

Application: Relevance Logic

Gentzen-style sequent calculus

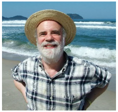
A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\;(\mathsf{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\;(\mathsf{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\;(\to_{\mathsf{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\;(\to_{\mathsf{R}})$$

Problem (provability)

Given a sequent $\Gamma \vdash A$, is it provable?



Theorem (Kripke 1959)

Provability is decidable in implicational relevance logic.

Application: Relevance Logic

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\,(\mathsf{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,(\mathsf{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\,(\to_{\mathsf{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to_{\mathsf{R}})$$

- subformula property
- irredundant proof searches
  - (C) and $(\to_{\mathsf{R}})$ commute: (C)'s only below a $(\to_{\mathsf{L}})$
  - rewrite proofs to apply (C) whenever possible
- irredundant proof branches are bad sequences for contraction
- ...which is wqo over the subformulæ of $\Gamma \vdash A$

Application: Relevance Logic

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\,(\mathsf{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,(\mathsf{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\,(\to_{\mathsf{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to_{\mathsf{R}})$$

- subformula property
- irredundant proof searches
  - (C) and ($\to_{\mathsf{R}}$) commute: (C)'s only below a ($\to_{\mathsf{L}}$)
  - rewrite proofs to apply (C) whenever possible
- irredundant proof branches are bad sequences for contraction
- …which is wqo over the subformulæ of $\Gamma \vdash A$

Application: Relevance Logic

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\ (\mathsf{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (\mathsf{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\ (\to_{\mathsf{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to_{\mathsf{R}})$$

- subformula property
- irredundant proof searches
  - (C) and $(\to_{\mathsf{R}})$ commute: (C)'s only below a $(\to_{\mathsf{L}})$
  - rewrite proofs to apply (C) whenever possible
- irredundant proof branches are bad sequences for contraction
- ...which is wqo over the subformulæ of $\Gamma \vdash A$

Application: Relevance Logic

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\ (\mathrm{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (\mathrm{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\ (\to_{\mathrm{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to_{\mathrm{R}})$$

- subformula property
- irredundant proof searches
  - (C) and $(\to_{\mathrm{R}})$ commute: (C)'s only below a $(\to_{\mathrm{L}})$
  - rewrite proofs to apply (C) whenever possible
- irredundant proof branches are bad sequences for contraction
- ...which is wqo over the subformulæ of $\Gamma \vdash A$

Application: Preservation Theorems

logic $\mathcal{L}$	example	$\hom_{\mathcal{L}}$
$\exists$FO	$\exists z.x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z)$	strong injective
$\exists$FO$^+(\neq)$	$\exists yy'.x \xrightarrow{R} y \wedge y' \xrightarrow{B} z \wedge y \neq y'$	injective
$\exists$FO$^+$	$\exists y.x \xrightarrow{G} y$	all

Fact

If $\psi \in \mathcal{L}$, $h \in \hom_{\mathcal{L}}$, *and* $D \models \psi(\mathbf{x})$, *then* $h(D) \models \psi(h(\mathbf{x}))$.

Application: Preservation Theorems

logic $\mathcal{L}$	example	$\mathrm{hom}_{\mathcal{L}}$
$\exists$FO	$\exists z.x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z)$	strong injective
$\exists$FO$^{+}(\neq)$	$\exists y y'.x \xrightarrow{R} y \wedge y' \xrightarrow{B} z \wedge y \neq y'$	injective
$\exists$FO^{+}	$\exists y.x \xrightarrow{G} y$	all

Definition

$D \leqslant_{\mathcal{L}} D'$ if $\exists h \in \mathrm{hom}_{\mathcal{L}}$ s.t. $D' = h(D)$.

Application: Preservation Theorems

logic $\mathcal{L}$	example	$\hom_{\mathcal{L}}$
$\exists$FO	$\exists z.x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z)$	strong injective
$\exists\text{FO}^{+}(\neq)$	$\exists yy'.x \xrightarrow{R} y \wedge y' \xrightarrow{B} z \wedge y \neq y'$	injective
$\exists\text{FO}^{+}$	$\exists y.x \xrightarrow{G} y$	all

Over arbitrary structures

Theorem (Łoś, Lyndon, Tarski)

If φ is an FO-sentence s.t. $[\![\varphi]\!]$ is upwards-closed for $\leqslant_{\mathcal{L}}$, then there exists $\psi \in \mathcal{L}$ with $[\![\varphi]\!] = [\![\psi]\!]$.

Application: Preservation Theorems

logic $\mathcal{L}$	example	$\hom_{\mathcal{L}}$
$\exists$FO	$\exists z.x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z)$	strong injective
$\exists\mathrm{FO}^{+}(\neq)$	$\exists y y'.x \xrightarrow{R} y \wedge y' \xrightarrow{B} z \wedge y \neq y'$	injective
$\exists\mathrm{FO}^{+}$	$\exists y.x \xrightarrow{G} y$	all

Over finite (relational) structures?

Application: Preservation Theorems

logic $\mathcal{L}$	example	$\hom_{\mathcal{L}}$
$\exists$FO	no [Tait 1959]	strong injective
$\exists\text{FO}^+(\neq)$	no [Ajtai & Gurevich 1994]	injective
$\exists\text{FO}^+$	yes [Rossman 2008]	all

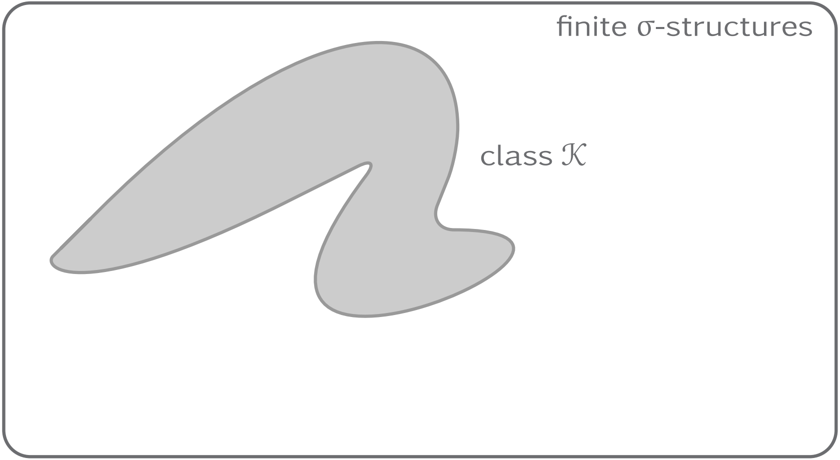
Over finite (relational) structures?

Application: Preservation Theorems

Over finite (relational) structures?

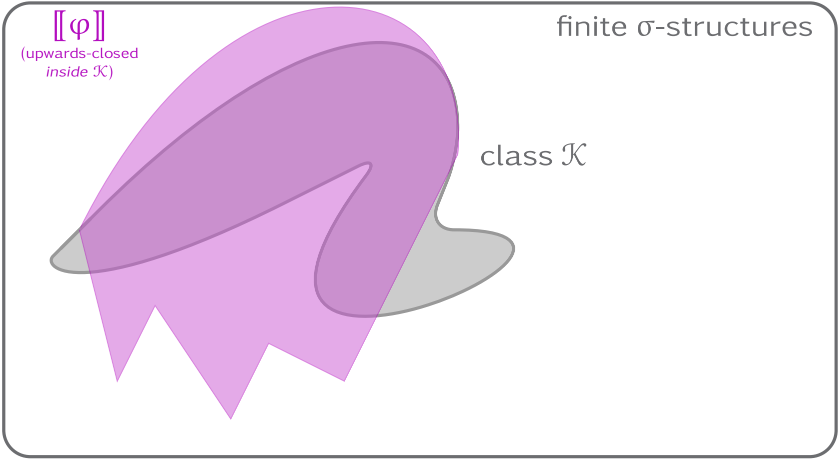
Application: Preservation Theorems

Over finite (relational) structures?



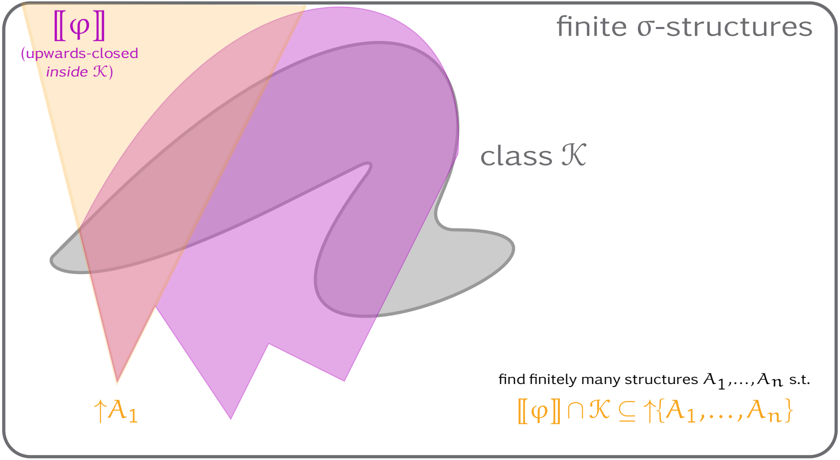
Application: Preservation Theorems

Over finite (relational) structures?



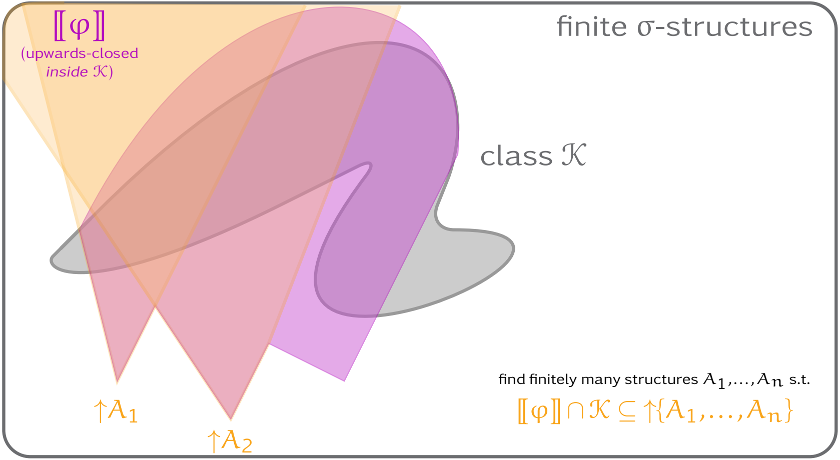
Application: Preservation Theorems

Over finite (relational) structures?



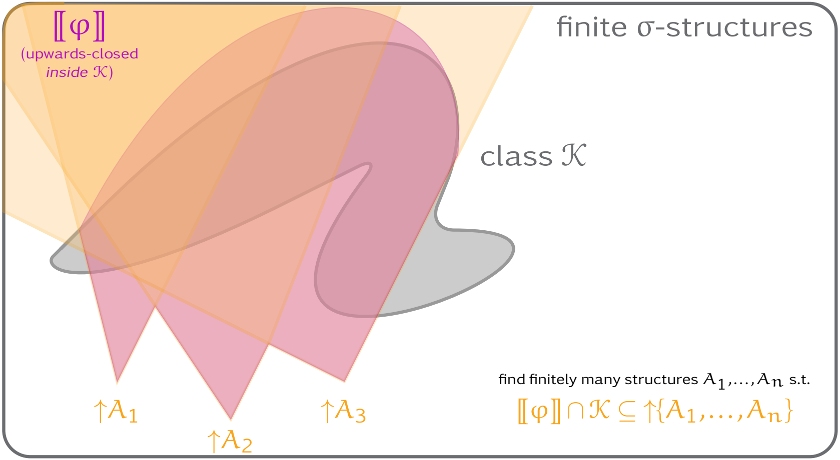
Application: Preservation Theorems

Over finite (relational) structures?



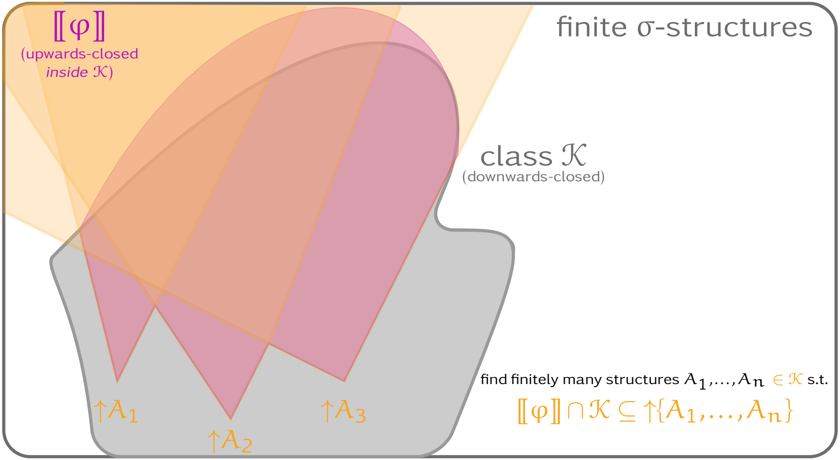
Application: Preservation Theorems

Over finite (relational) structures?



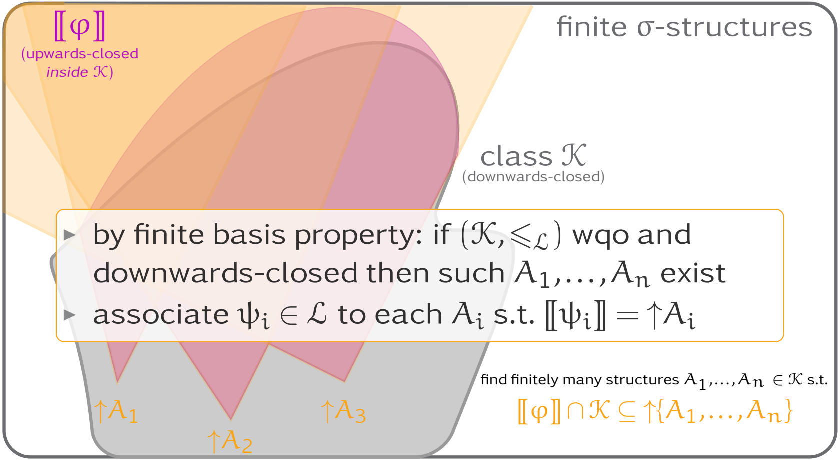
Application: Preservation Theorems

Over finite (relational) structures?



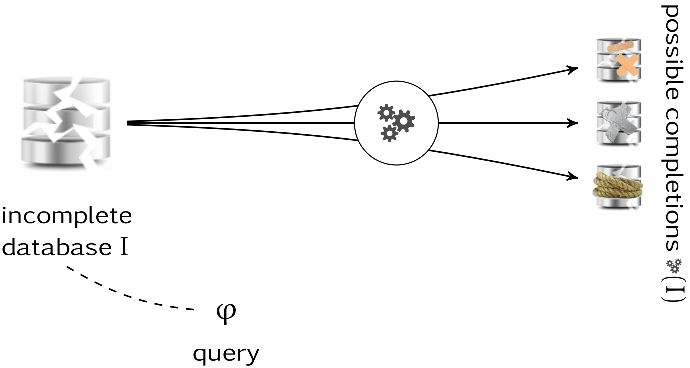
Application: Preservation Theorems

Over finite (relational) structures?

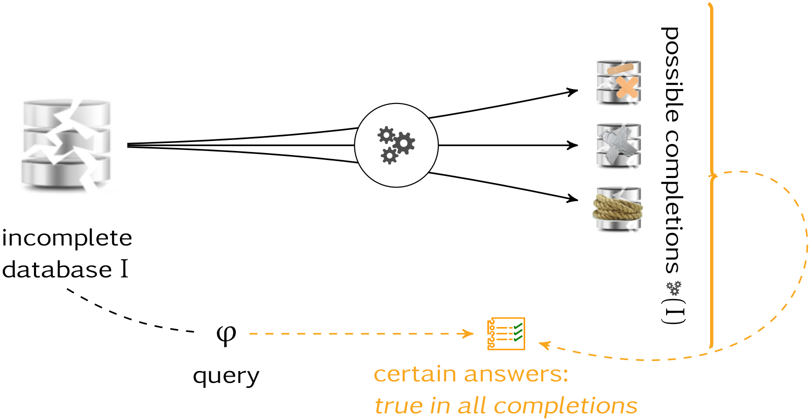


- by finite basis property: if $(\mathcal{K}, \leqslant_{\mathcal{L}})$ wqo and downwards-closed then such $A_1, \ldots, A_n$ exist
- associate $\psi_i \in \mathcal{L}$ to each A_i s.t. $[\![\psi_i]\!] = \uparrow A_i$

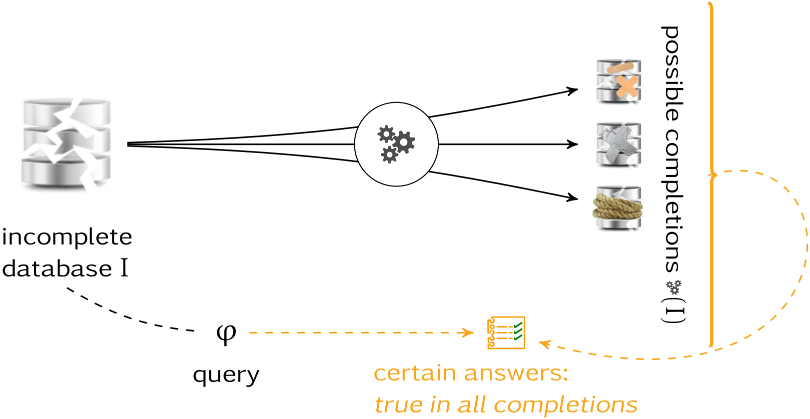
Application: Certain Answers



Application: Certain Answers

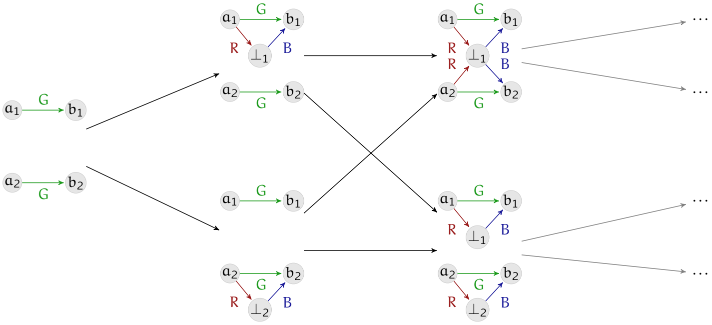


Application: Certain Answers

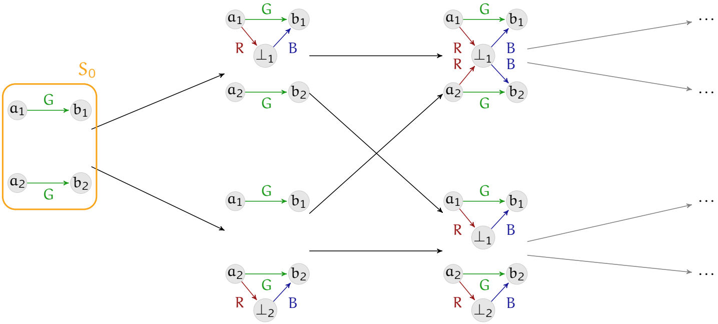


$$\mathrm{certain}_I(\varphi) = \bigcap_{D \in ⚙(I)} \{\mathbf{x} \mid D \models \varphi(\mathbf{x})\}$$

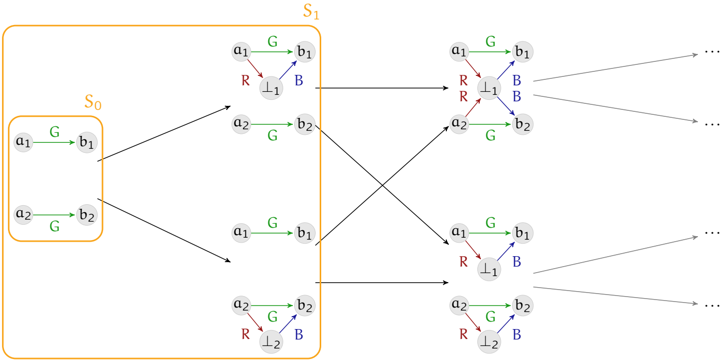
Chase of $x \xrightarrow{G} z \implies \exists y\,.\,x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ for $\varphi \in \exists\mathrm{FO}^{+}(\neq)$



Chase of $x \xrightarrow{G} z \implies \exists y \,.\, x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ for $\varphi \in \exists\mathrm{FO}^+(\neq)$

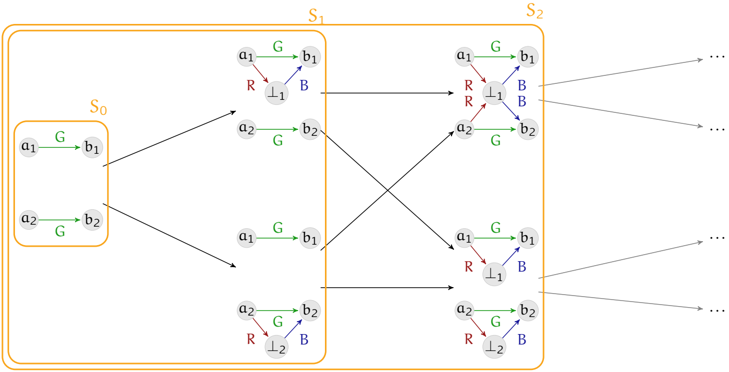


Chase of $x \xrightarrow{G} z \implies \exists y \,.\, x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ for $\varphi \in \exists\mathrm{FO}^+(\neq)$



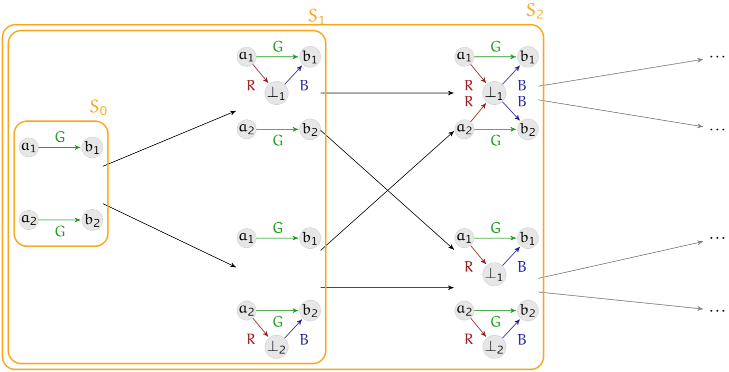
$\uparrow S_0 \quad \subseteq \quad \uparrow S_1$

Chase of $x \xrightarrow{G} z \implies \exists y \,.\, x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ for $\varphi \in \exists\mathrm{FO}^+(\neq)$



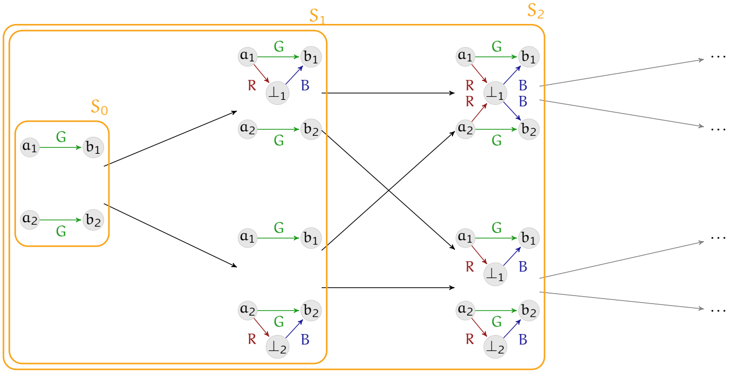
$\uparrow S_0 \quad \subseteq \quad \uparrow S_1 \quad \subseteq \quad \uparrow S_2$

Chase of $x \xrightarrow{G} z \implies \exists y \,.\, x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ for $\varphi \in \exists\mathrm{FO}^+(\neq)$



$\uparrow S_0 \subseteq \uparrow S_1 \subseteq \uparrow S_2 \equiv \cdots$

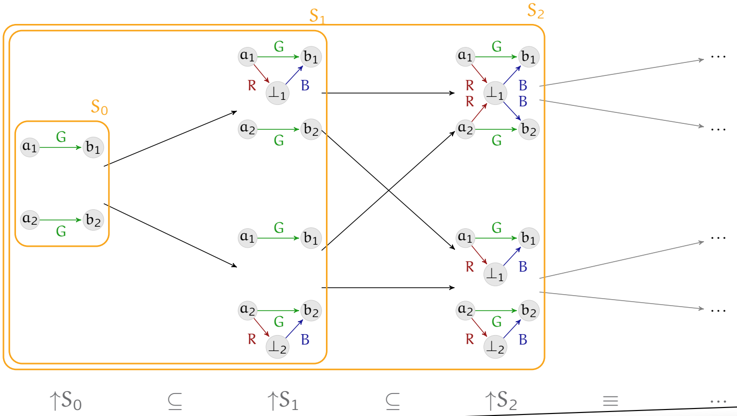
Chase of $x \xrightarrow{G} z \implies \exists y \,.\, x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ for $\varphi \in \exists\mathrm{FO}^+(\neq)$



$\uparrow S_0 \quad \subseteq \quad \uparrow S_1 \quad \subseteq \quad \uparrow S_2 \quad \equiv \quad \cdots$

- over a wqo: by ascending chain condition, $\uparrow S_0 \subseteq \uparrow S_1 \subseteq \cdots$ always stabilises to $\uparrow S_\star$
- $\mathrm{certain}_I(\varphi) = (\mathrm{dom}\, I)^* \cap \bigcap_{B \in S_\star} \{\mathbf{x} \mid B \models \varphi(\mathbf{x})\}$

CHASE OF $x \xrightarrow{G} z \implies \exists y\,.\,x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ FOR $\varphi \in \exists FO^{+}(\neq)$



- over a wqo: by ascending chain condition, $\uparrow S_0 \subseteq \uparrow S_1 \subseteq \cdots$ always stabilises to $\uparrow S_\star$
- $\text{certain}_I(\varphi) = (\text{dom}\, I)^* \cap \bigcap_{B \in S_\star} \{\mathbf{x} \mid B \models \varphi(\mathbf{x})\}$

Research Projects

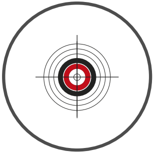
- Perspectives of a doctoral thesis
- Job applications
- Funding applications
- For yourselves

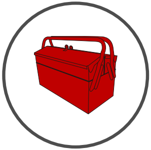
Research Projects

- Perspectives of a doctoral thesis
- Job applications
- Funding applications
- For yourselves

Research Drives

Problems

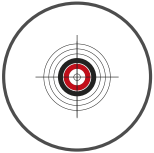


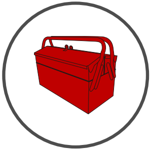


Foundations

Research Drives

Problems

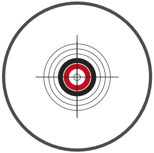




Foundations

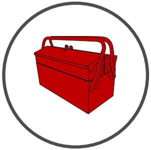
- attainable?
- specialised?

Research Drives



Problems

- attainable?
- specialised?



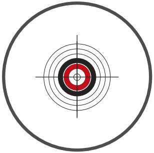
Foundations

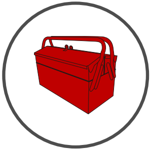
- applications?
- impact?

Research Drives

Problems

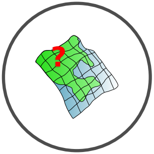
- attainable?
- specialised?





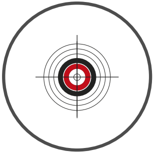
Foundations

- applications?
- impact?



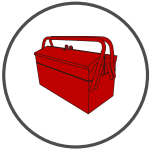
Curiosity

Research Drives



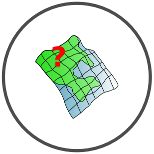
Problems

- attainable?
- specialised?



Foundations

- applications?
- impact?



Curiosity

- misrepresentation?

Human Dimension

Will you collaborate on your project?

- locally at your institution?
- internationally?
- by supervising students?
- by animating a local seminar?
- by organising specialised workshops?
- by giving lectures locally or at summer schools?

Human Dimension

Will you collaborate on your project?

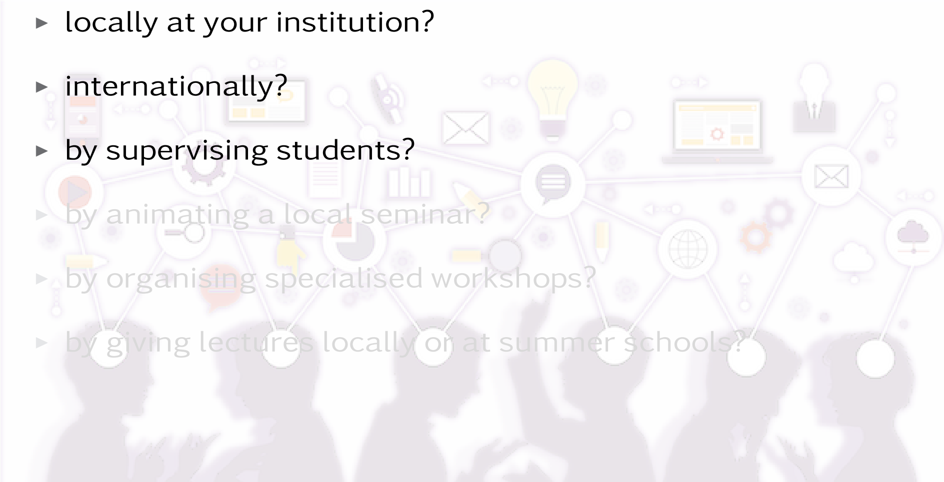
- locally at your institution?
- internationally?
- by supervising students?
- by animating a local seminar?
- by organising specialised workshops?
- by giving lectures locally or at summer schools?

Human Dimension

Will you collaborate on your project?

- locally at your institution?
- internationally?
- by supervising students?
- by animating a local seminar?
- by organising specialised workshops?
- by giving lectures locally or at summer schools?

Human Dimension



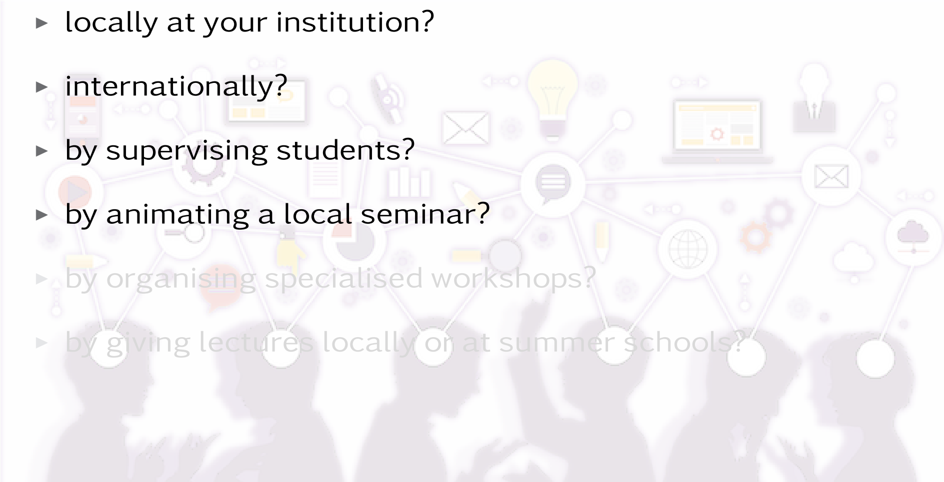
Will you collaborate on your project?

- locally at your institution?
- internationally?
- by supervising students?
- by animating a local seminar?
- by organising specialised workshops?
- by giving lectures locally or at summer schools?

Human Dimension

Will you collaborate on your project?

- locally at your institution?
- internationally?
- by supervising students?
- by animating a local seminar?
- by organising specialised workshops?
- by giving lectures locally or at summer schools?



Human Dimension

Will you collaborate on your project?

- locally at your institution?
- internationally?
- by supervising students?
- by animating a local seminar?
- by organising specialised workshops?
- by giving lectures locally or at summer schools?

Human Dimension

Will you collaborate on your project?

- locally at your institution?
- internationally?
- by supervising students?
- by animating a local seminar?
- by organising specialised workshops?
- by giving lectures locally or at summer schools?