Fixed Point Logics on Hemimetric Spaces

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The perfect core

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Theorem (General C-B)

If X is any topological space and $A \subseteq X$, A has a maximal perfect subset, called its **perfect core.**

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- $\Diamond\varphi := \neg\Box\neg\varphi$

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Kripke semantics: Models are triples $(W, \sqsubset, [\![\cdot]\!])$.

$$w \in [\![\Box\varphi]\!] \Leftrightarrow \forall v \sqsupset w (v \in [\![\varphi]\!]).$$

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Example: Transitive closure:

$$\Diamond^*\varphi := \mu p.(\varphi \vee \Diamond p)$$

Topological semantics

If X is a topological space and $A \subseteq X$, define the **Cantor derivative** or **set of limit points of** A by

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The semantics of the full μ-calculus is well-defined over any topological space.

Hemimetric spaces

A *hemimetric on* a set X is a function $\vartriangle\colon X \times X \to \mathbb{R}_{\geq 0}$ such that, for all $x, y, z \in X$:

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- If $\vartriangle$ is is symmetric in the sense that $\vartriangle(x, y) = \vartriangle(y, x)$, then $\vartriangle$ is a *metric.*

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Theorem

Let φ be a formula of the μ-calculus. Then, the following are equivalent:

1. *φ is valid over the class of all Hausdorff (more generally, T_D) spaces*

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Essentially due to Bezhanishvili and Lucero-Bryan, 2012, combined with Goldblatt and Hodkinson, 2017. Extends McKinsey and Tarski, 1944.

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3. φ *is valid over the class of all hemimetric spaces [F-D and Gougeon, LICS'23]*

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2. *φ is valid over the class of all finite, weakly transitive frames where each cluster has at most one irreflexive point [Baltag et al., LICS'21]*
3. *φ is valid over the class of all quasimetric spaces [F-D and Gougeon, LICS'23]*

The tangled derivative

Define

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1. If $(W, \sqsubset)$ is a finite, transitive frame and $A_1, \ldots, A_n \subseteq W$, $w \in [\![\Diamond^{\infty}\{A_1, \ldots, A_n\}]\!]$ iff there is a reflexive cluster $\mathcal{C} \sqsupseteq w$ such that for each $i \leq n$, $A_i \cap \mathcal{C} \neq \varnothing$.

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2. Topologically, $[\![\Diamond^{\infty}\{A_1, \ldots, A_n\}]\!]$ is the largest subspace in which every A_i is dense.

Universality of tangle

Theorem (Dawar and Otto 2009)

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Corollary

Every formula of the μ-calculus is equivalent to a formula in $\mathcal{L}^{\Diamond}_{\Diamond\infty}$ over the class of metric spaces.

Expressive incompleteness

- Tangled derivative: $\Diamond^{\infty}$
- Tangled closure: $\diamonddot^{\infty}$ (analogous, but defined in terms of the ‘reflexive’ closure operator)

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Given $\Phi = (\varphi_1, \ldots, \varphi_n)$,

$$\blacklozenge^\infty \Phi := \nu p. \bigvee_{i \leq n} \Big(\odot (\varphi_i \wedge \blacklozenge^\infty \Phi) \wedge \bigwedge_{j \neq i} \Diamond(\varphi_j \wedge \blacklozenge^\infty \Phi) \Big)$$

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Thank you!