Fixed Point Logics on Hemimetric Spaces

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LICS’23
The perfect core

Theorem (Cantor-Bendixson)

*Any closed subset of a Polish space is the disjoint union of a perfect set and a countable set.*
The perfect core

Theorem (Cantor-Bendixson)

Any closed subset of a Polish space is the disjoint union of a perfect set and a countable set.

Theorem (General C-B)

If $X$ is any topological space and $A \subseteq X$, $A$ has a maximal perfect subset, called its perfect core.
Unimodal language

Modal language $\mathcal{L}^\Diamond$:

$p \mid \neg \varphi \mid \varphi \land \psi \mid \Box \varphi$
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$p \mid \neg \phi \mid \phi \land \psi \mid \Box \phi$

Usual abbreviations:

$\lor \phi \lor \psi := \neg (\neg \phi \land \neg \psi)$

$\Diamond \phi := \neg \Box \neg \phi$
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Usual abbreviations:

- $\varphi \lor \psi := \neg (\neg \varphi \land \neg \psi)$
- $\Diamond \varphi := \neg \Box \neg \varphi$

Kripke semantics: Models are triples $(W, \Box, [\cdot])$.

$$w \in [\Box \varphi] \iff \forall v \Box w (v \in [\varphi]).$$
The $\mu$-calculus

Language $\mathcal{L}^\Diamond_\mu$: Add expressions $\mu p.\varphi(p)$ to the modal language, where $p$ appears only positively in $\varphi$. 
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Language $L^\diamondsuit_\mu$:
Add expressions $\mu p. \varphi(p)$ to the modal language, where $p$ appears only positively in $\varphi$.

$\mu p. \varphi(p)$ is the least fixed point of $A \mapsto [\varphi(A)]$.

Example: Transitive closure: $\Box^* \varphi := \mu p. (\varphi(p) \lor \Box p)$
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Language $\mathcal{L}_\mu^\Diamond$:
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- $[\mu p.\varphi(p)]$ is the least fixed point of $A \mapsto [\varphi(A)]$.

- $\nu p.\varphi(p) := \neg \mu p.\neg \varphi(\neg p)$ is the greatest fixed point of $A \mapsto [\varphi(A)]$. 

Example:
Transitive closure: $\mathcal{L}_\mu^\Diamond$.
The $\mu$-calculus

Language $\mathcal{L}^\diamondsuit_\mu$:
Add expressions $\mu p. \varphi(p)$ to the modal language, where $p$ appears only \textbf{positively} in $\varphi$.

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Example: Transitive closure:

$$\diamondsuit^* \varphi := \mu p. (\varphi \lor \diamondsuit p)$$
Topological semantics

If $X$ is a topological space and $A \subseteq X$, define the **Cantor derivative** or **set of limit points of** $A$ by

$$dA = \left\{ x \in X : x \in c(A \setminus \{x\}) \right\}.$$
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**d-Semantics:** $[\Diamond \varphi] := d[\varphi]$. 

**Weak transitivity axiom:** $\varphi \wedge \Box \varphi \rightarrow \Box \Box \varphi$. 

The semantics of the full $\mu$-calculus is well-defined over any topological space.
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Hemimetric spaces

A hemimetric on a set $X$ is a function $\Delta : X \times X \to \mathbb{R}_{\geq 0}$ such that, for all $x, y, z \in X$:

1. $\Delta(x, x) = 0$,

2. $\Delta(x, z) \leq \Delta(x, y) + \Delta(y, z)$. 

If $\Delta$ is such that $\Delta(x, y) + \Delta(y, x) = 0$ implies $x = y$, then $\Delta$ is a quasimetric.

If $\Delta$ is symmetric in the sense that $\Delta(x, y) = \Delta(y, x)$, then $\Delta$ is a metric.
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The derivative $dA$ of $A \subseteq X$ is the set of all $x \in X$ such that $A \setminus \{x\}$ contains a sequence of points $(x_n)_{n \in \mathbb{N}}$ such that $\Delta(x, x_n) \to 0$. 
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Metric completeness theorems

Theorem

Let $\varphi$ be a formula of the $\mu$-calculus. Then, the following are equivalent:

1. $\varphi$ is valid over the class of all Hausdorff (more generally, $T_D$) spaces
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1. \( \varphi \) is valid over the class of all topological spaces
2. \( \varphi \) is valid over the class of all finite, weakly transitive frames [Baltag et al., LICS’21]
3. \( \varphi \) is valid over the class of all hemimetric spaces [F-D and Gougeon, LICS’23]
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Quasimetric completeness theorem

**Theorem**

Let \( \varphi \) be a formula of the \( \mu \)-calculus. Then, the following are equivalent.

1. \( \varphi \) is valid over the class of all \( T_0 \) spaces (different points have different sets of neighborhoods)
2. \( \varphi \) is valid over the class of all finite, weakly transitive frames where each cluster has at most one irreflexive point [Baltag et al., LICS'21]
3. \( \varphi \) is valid over the class of all quasimetric spaces [F-D and Gougeon, LICS'23]
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The tangled derivative

Define

$$\Diamond^\infty \{\varphi_1, \ldots, \varphi_n\} := \nu p. \bigwedge \Diamond (p \land \varphi_i).$$
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1. If \((W, \square)\) is a finite, transitive frame and \(A_1, \ldots, A_n \subseteq W\), \(w \in \left[ \diamond \infty \{ A_1, \ldots, A_n \} \right]\) iff there is a reflexive cluster \(C \sqsupseteq w\) such that for each \(i \leq n\), \(A_i \cap C \neq \emptyset\).
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\[ \diamondsuit^\infty \{ \varphi_1, \ldots, \varphi_n \} := \nu p. \bigwedge \diamondsuit (p \land \varphi_i) . \]

1. If \((W, \sqsubseteq)\) is a finite, transitive frame and \(A_1, \ldots, A_n \subseteq W\), \(w \in \llbracket \diamondsuit^\infty \{ A_1, \ldots, A_n \} \rrbracket\) iff there is a reflexive cluster \(C \subseteq w\) such that for each \(i \leq n\), \(A_i \cap C \neq \emptyset\).

2. Topologically, \(\llbracket \diamondsuit^\infty \{ A_1, \ldots, A_n \} \rrbracket\) is the largest subspace in which every \(A_i\) is dense.
Universality of tangle

Theorem (Dawar and Otto 2009)

Every formula of the $\mu$-calculus is equivalent to a formula in $\mathcal{L}^{\Diamond}^{\infty}$ over the class of transitive frames.
Universality of tangle

Theorem (Dawar and Otto 2009)

Every formula of the $\mu$-calculus is equivalent to a formula in $\mathcal{L}^\dagger_\infty$ over the class of transitive frames.

Corollary

Every formula of the $\mu$-calculus is equivalent to a formula in $\mathcal{L}^\dagger_\infty$ over the class of metric spaces.
Expressive incompleteness

- Tangled derivative: ◇∞
- Tangled closure: ◇∞ (analogous, but defined in terms of the ‘reflexive’ closure operator)

Theorem 1. ◇∞ is not definable in L♢ over the class of Tデン spaces

Theorem 2. ◇∞ is not definable in L♢♢∞ over the class of Tデン spaces [Baltag et al.]
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1. $\diamondsuit^\infty$ is not definable in $\mathcal{L}\diamondsuit^\infty$ over the class of $T_D$ spaces [Baltag et al.]
Expressive incompleteness

- Tangled derivative: $\diamondsuit \infty$
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Theorem

1. $\diamondsuit \infty$ is not definable in $\mathcal{L}^{\diamondsuit \infty}$ over the class of $T_D$ spaces [Baltag et al.]
2. $\diamondsuit \infty$ is not definable in $\mathcal{L}^{\diamond \infty}$ over the class of $T_0$ spaces [F-D, Gougeon]
The hybrid tangle

Given $\Phi = (\varphi_1, \ldots, \varphi_n)$,

$$\Diamond^\infty \Phi := \nu p. \bigvee_{i \leq n} \left( \Diamond (\varphi_i \land \Diamond^\infty \Phi) \land \bigwedge_{j \neq i} \Diamond (\varphi_j \land \Diamond^\infty \Phi) \right)$$
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Given $\Phi = (\varphi_1, \ldots, \varphi_n)$,

$$\blacklozenge^\infty \Phi := \nu p. \bigvee_{i \leq n} \left( \blacklozenge (\varphi_i \land \blacklozenge^\infty \Phi) \land \bigwedge_{j \neq i} \blacklozenge (\varphi_j \land \blacklozenge^\infty \Phi) \right)$$

Theorem (F-D, Gougeon)

$\blacklozenge^\infty$ and $\blacklozenge^\infty$ are definable in $\mathcal{L}_{\blacklozenge^\infty}$ over the class of topological spaces.
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Conclusion

- Metric completeness theorems for modal logic can be extended to arbitrary topological spaces by considering hemimetric spaces.
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