Taylor Expansion as a Monad in Models of DiLL
Codigging for Differential Categories

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TODAY'S STORY: Codigging for Differential Categories

What is Codigging?

In MELL there are four main structural rules (+ functorial promotion):

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \textbf{Dereliction} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \textbf{Contraction} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \textbf{Weakening} \qquad \frac{\Gamma, !!A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \textbf{Digging}$$

In DILL we add three structural rules:

$$\frac{\Gamma, !A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textbf{Codereliction} \qquad \frac{\Gamma, !A \vdash B}{\Gamma, !A, !A \vdash \Delta}\ \textbf{Cocontraction} \qquad \frac{\Gamma, !A \vdash \Delta}{\Gamma \vdash \Delta}\ \textbf{Coweakening}$$

T. Ehrhard, L. Regnier **Differential interaction nets**

T. Ehrhard **An introduction to Differential Linear Logic: proof-nets, models and antiderivatives**

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But it's not exactly symmetric: missing **CODIGGING**.

Today's story: adding codigging to DILL and we will explain that:

- codigging $\equiv$ Exponential e^x
- Codigging Axioms $\equiv$ Taylor Expansion
- DILL + codigging $\Rightarrow$ Quantitative, i.e., proofs are interpreted by **power series**
- codigging tells us that every proof is equal to its Taylor series.

Differential Storage Category

To help understand codigging we take the categorical point of view. So a categorical model of DILL is a **differential storage category** which in particular is:

- Symmetric monoidal category with tensor product $\otimes$ and unit I
- Equipped with a comonad $! : \mathbb{X} \to \mathbb{X}$ with $\underbrace{!A \xrightarrow{d_A} A}_{\textbf{Dereliction}}$ and $\underbrace{!A \xrightarrow{\mathsf{p}_A} !!A}_{\textbf{Digging}}$
- $!A$ is a bicommutative bimonoid with:
  - Comonoid structure $\underbrace{!A \xrightarrow{c_A} !A \otimes !A}_{\textbf{Contraction}}$ and $\underbrace{!A \xrightarrow{w_A} I}_{\textbf{Weakening}}$
  - Monoid structure $\underbrace{!A \otimes !A \xrightarrow{\overline{c}_A} !A}_{\textbf{Cocontraction}}$ and $\underbrace{I \xrightarrow{\overline{w}_A} !A}_{\textbf{Coweakening}}$
- Differential part: $\underbrace{A \xrightarrow{\overline{d}_A} !A}_{\textbf{Codereliction}}$

such that various identities hold:

R. Blute, R. Cockett, R.A.G. Seely, JS P. Lemay **Differential categories revisited**.

Codigging

In a differential storage category, a **codigging** is a natural transformation:

$$!!A \xrightarrow[\textbf{Codigging}]{\overline{\mathsf{p}}_A} !A$$

What should the axioms be?

They should be dual of those of $!A \xrightarrow[\textbf{Digging}]{\mathsf{p}_A} !!A$

- $(!, \mathsf{p}, \mathsf{d})$ is a comonad, so $(!, \overline{\mathsf{p}}, \overline{\mathsf{d}})$ should be a monad
- p is a comonoid morphism, so $\overline{\mathsf{p}}$ should be a monoid morphism
- p and $\overline{\mathsf{d}}$ are compatible via a chain rule axiom, so $\overline{\mathsf{p}}$ and d should satisfy the dual of the chain rule.

Codigging is a Generalized Exponential Function

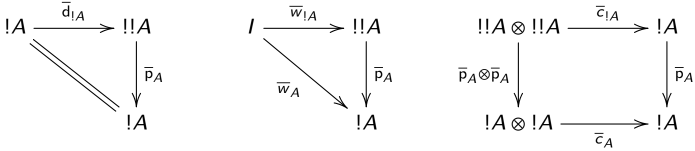
- Exponential functions e^x where generalized in the context of differential categories:

  JS P. Lemay **Exponential Functions for Cartesian Differential Categories**.

- For a differential storage category, a generalized exponential function for the object A correspond to a map $!A \xrightarrow{E} A$ which satisfies three axioms based on:

$$(e^x)' = e^x \qquad e^0 = 1 \qquad e^{x+y} = e^x e^y$$

- The axioms of codigging tell us that $!!A \xrightarrow{\overline{\mathsf{p}}_A} !A$ is an generalized exponential function on $!A$. This means these diagrams commute:



Distribution and the Convolutional Exponential

In Classical DiLL, elements of $!A$ can be interpreted as *distributions*: linear scalar maps acting on non-linear maps, $[\![!A]\!] := [[\![A]\!], I] \multimap I$.

Often it is sufficient to define what a non-linear map does on dirac distributions $\delta_x : f \mapsto f(x)$.

Since $\overline{\mathsf{p}}$ is a generalization of e^x, it will be useful to use a very naive "illicit formula" for $\overline{\mathsf{p}}$ based on the exponential function's power series:

$$e^x = \sum_n \frac{x^n}{n!}$$

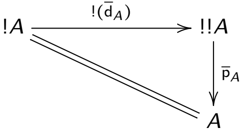
Assuming that we have proper convergences and can operate scalar multiplication by rationals, we obtain the following formula for codigging:

$$\overline{\mathsf{p}}_A : \delta_\phi \mapsto \exp^*(\phi) = \sum_n \frac{\phi^{*^n}}{n!}$$

where $*$ is the convolution of distributions (which is what the cocontraction is), and the above formula is called the convolutional exponential.

Monad Axiom = Taylor Expansion

One of the monad axioms is that:



On dirac distributions, the codereliction gives the differential operator at zero:

$$\bar{\mathrm{d}}_A : \delta_x \mapsto D_0(\_)(x)$$

So we get:

$$\delta_x = \sum_n \frac{D_0^{(n)}(\_)(x)}{n!}$$

where $D_0^{(n)}$ is the distribution mapping a function to its n-th differential at 0.

Then for a non-linear map $f : !A \to B$, we get that

$$f(x) = \sum_n \frac{D_0^{(n)}(f)(x)}{n!}$$

In other words, in a model with codigging, every non-linear map is equal to its Taylor expansion at 0. This implies that any model of DiLL with codigging needs to be a *quantitative model*, with non-linear maps being power series, such that the exponential function series also converges.

Other Axioms

The remaining two axioms are:

- Associativity:

$$
\begin{array}{ccc}
!!!A & \xrightarrow{\overline{\mathsf{p}}_{!A}} & !!A \\
{\scriptstyle !(\overline{\mathsf{p}}_A)}\downarrow & & \downarrow{\scriptstyle \overline{\mathsf{p}}_A} \\
!!A & \xrightarrow[\overline{\mathsf{p}}_A]{} & !A
\end{array}
$$

  How to interpret the exponential of the exponential, e^{e^x}.

- Coherence with dereliction: dual of the chain rule.

$$
\begin{array}{ccccccc}
!!A & & \xrightarrow{\overline{\mathsf{p}}_A} & & & & !A \\
{\scriptstyle \mathsf{c}_{!A}}\downarrow & & & & & & \downarrow{\scriptstyle d_A} \\
!!A \otimes !!A & \xrightarrow[\overline{\mathsf{p}}_A \otimes \mathsf{d}_{!A}]{} & !A \otimes !A & \xrightarrow[\mathsf{w}_A \otimes \mathsf{d}_A]{} & I \otimes A & \xrightarrow[\cong]{} & A
\end{array}
$$

  precisely tells us that:

$$\sum_n \frac{nx}{n!} = e^1 x.$$

Examples

Example

- Relational model where $!X$ is the finite multisets over X and the codigging $\overline{\mathsf{p}}_X \subseteq !!X \times !X$ relates a finite multiset of finite multisets to its disjoint union:

$$\overline{\mathsf{p}}_X \coloneqq \{([m_1, \cdots, m_n], m_1 \sqcup \cdots \sqcup m_n) \mid \forall n \in \mathbb{N}, m_i \in !X\}$$

Example

- General construction given by:

Laird, J. and Manzonetto, G. and McCusker, G., **Constructing differential categories and deconstructing categories of games**

So from any symmetric monoidal category, we can construct a model with codigging.

Example

- Quantum related examples

Vicary, J. **A categorical framework for the quantum harmonic oscillator**

Pagani, M. and Selinger, P. and Valiron, B., **Applying Quantitative Semantics to Higher-Order Quantum Computing**

Example

- Non-Example: unfortunately finiteness spaces or Köthe spaces don't have codigging.

Other Interesting Results:

- We make precise the "illicit formula" for codigging by introducing a metric for differential categories in which:

$$\overline{\mathsf{p}}_A : \delta_\phi \mapsto \exp^*(\phi) = \sum_n \frac{\phi^{*^n}}{n!}$$

  converges properly. We call such a setting a **Taylor differential category**, and this metric is compatible with any other established notion of infinite sum or convergence (like topological metric) that was already there.

- The algebras of the monad $(!, \overline{\mathsf{p}}, \overline{\mathsf{d}})$ are commutative monoids equipped with a generalized exponential function e^x.

- Provide a graded and polarized model of DiLL with codigging (solving the issue with finiteness spaces and Köthe spaces).

Conclusion:

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- Codigging Axioms ≡ Taylor Expansion
- DILL + codigging ⇒ Quantitative, i.e., proofs are interpreted by **power series**
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Future Work:

- Find more examples of codigging
- Differential Proof-Nets with codigging
- Compatibility between digging p and codigging $\overline{\mathrm{p}}$, potentially given by a mixed distributive law $\lambda_A : !!A \to !!A$
- What does codigging add to λ-terms and the resource calculi?

HOPE YOU ENJOYED MY TALK! If found it interesting please go read our paper.

THANKS FOR LISTENING!

MERCI!

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