The cartesian closed bicategory of thin spans

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Witnesses

This work: a new **proof-relevant** version of the **relational model** of linear logic.
proof-relevant $\rightsquigarrow$ computational behaviors described more accurately.

Witnesses in plain relational model: pairs (input,output)
Witnesses in a p.-r. model: triples (input,output,reason)

Example with a non-deterministic operator $\circledvee$:

$$x : \mathbf{Nat} \vdash (x - 2) \circledvee (x \times 2) : \mathbf{Nat}$$

plain witnesses	**p.-r. witnesses**
$([x \mapsto n], n-2)$, $([x \mapsto n], 2n)$	$([x \mapsto n]$,n-2,$\circledvee$-left), $([x \mapsto n]$,2n,$\circledvee$-right)

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Example with a non-deterministic operator $\oslash$:

$$\vdash \text{"same"} \oslash \text{"same"} : \textbf{string}$$

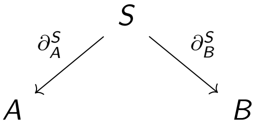
plain witnesses	**p.-r. witnesses**
($\emptyset$,"same")	($\emptyset$,"same",$\oslash$-left), ($\emptyset$,"same",$\oslash$-right)

Spans

Witnesses of executions of a program $x : A \vdash s : B$: set of triples

$$S = \{ (a_i, b_i, r_i) \mid i \in \mathcal{I} \}.$$

Canonical projections from S to A and B, which forms a **span**



Composition

What about composition?

$$x : A \vdash p : B \qquad\qquad y : B \vdash q : C$$

$$S = \{\, (a_i, b_i, r_i) \mid i \in \mathcal{I} \,\} \qquad T = \{\, (b'_j, c_j, s_j) \mid j \in \mathcal{J} \,\}.$$

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Witnesses of "$q \circ p$":

$$T \odot S = \{ (a_i, c_j, (r_i, s_j)) \quad \mid \quad (a_i, b_i, r_i) \in S, \quad (b'_j, c_j, s_j) \in T, \quad b_i = b'_j \}$$

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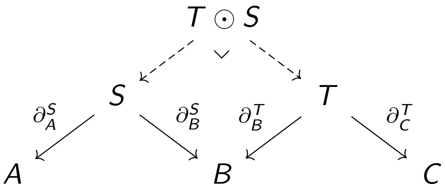

Towards a computational model

Spans can be used to describe the semantics of toy examples and compose them.

But what about more general examples with

- lambda abstractions and higher-order functions,
- multiple uses of variables (CBN setting).

To handle this: we introduce an **exponential modality** ! and consider spans of the form

$$A \leftarrow S \rightarrow B \quad \rightsquigarrow \quad !A \leftarrow S \rightarrow B$$

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Exponential modality

For **Span**, we are led to define

$$!A \quad = \quad \mathbf{Fam}(A)$$

where $(a_i)_{i \in I} \in \mathbf{Fam}(A)$ are finite families of elements of A.

Problem: two families $(a_i)_{i \in I}$ and $(a'_i)_{i \in I'}$ of $\mathbf{Fam}(A)$ can be equivalent up to re-indexing, but we **cannot express that** with spans of sets.

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Exponential modality

Solution: work with spans of ~~sets~~ **groupoids**.

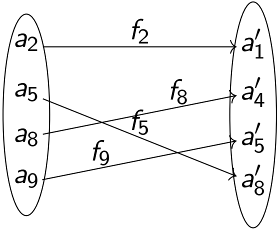
$\mathbf{Fam}(A)$ is now a groupoid with morphisms $(a_i)_{i \in I} \to (a'_i)_{i \in I'} \in \mathbf{Fam}(A)$ defined as pairs

$$(\pi, (f_i)_{i \in I})$$

with

$$\pi \colon I \xrightarrow{\sim} I' \qquad \text{and} \qquad f_i \colon a_i \to a'_{\pi(i)} \in A \text{ for every } i \in I.$$

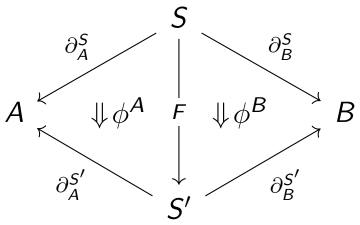
Graphically,



Morphisms of spans

In order to relate equivalent-up-to-re-indexing spans, we introduce **morphisms** between them.

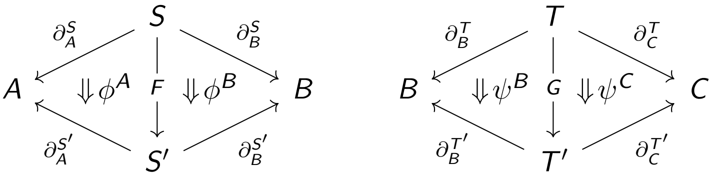
A morphism between two spans S and S' is the data of (F, ϕ^A, ϕ^B) as in



We get a **2-dimensional structure** with groupoids as 0-cells, spans as 1-cells and morphisms of spans as 2-cells.

Morphisms of spans

We must now give a definition for the horizontal composition of these morphisms of spans, a.k.a. the **2-cells** of our model.



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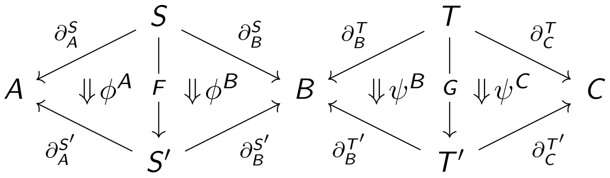

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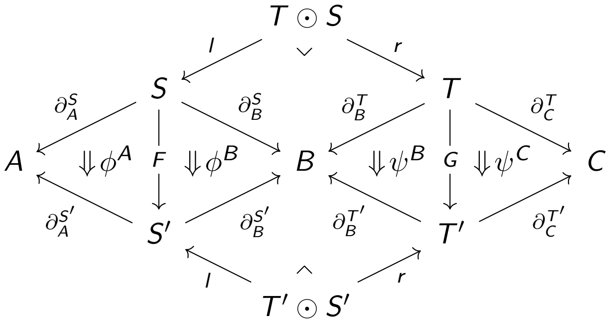

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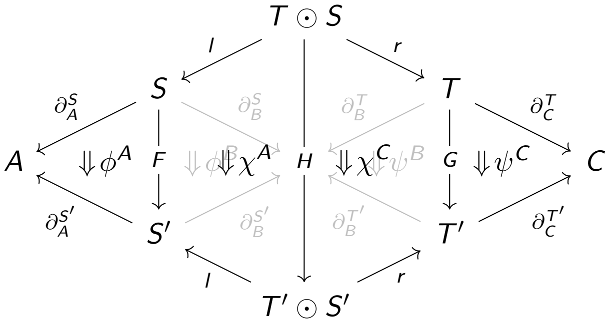

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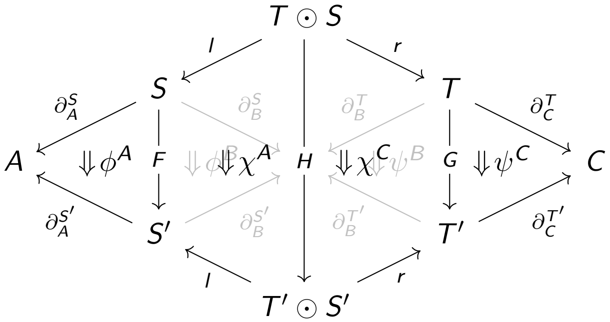


The above construction needs **bipullbacks** instead of pullbacks to be well-defined.

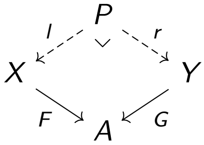
But bipullbacks are less easily computable than pullbacks. How to stick to the latter?

Uniformity

A uniform groupoid $\mathcal{A} = (A, \mathcal{U}_{\mathcal{A}})$ is the data of

- a groupoid A
- a class $\mathcal{U}_{\mathcal{A}}$ of functors $F\colon X \to A$, such that $\mathcal{U}_{\mathcal{A}}^{\perp\perp} = \mathcal{U}_{\mathcal{A}}$

with $F \perp G$ when the pullback



is a bipullback.

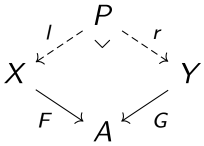
Unif: **uniform groupoids** as 0-cells, **"uniform spans"** as 1-cells, morphism of spans as 2-cells.

Result: uniformity allows composition **with plain pullbacks** of uniform spans!

Thinness

Problem: **Unif** is not a bicategory, because horizontal composition is still non-canonical.

Solution: another layer of orthogonality $\perp\!\!\!\perp$ (the **thin** one) which ensures canonicity:
$F \perp\!\!\!\perp G$ when the pullback



is discrete.

Thin: **thin groupoids** as 0-cells, **“thin spans”** as 1-cells, morphism of spans as 2-cells. (roughly)

Theorem (Clairambault, F.)

Thin *is a bicategory.*

Cartesian closure

Recall the definition of the exponential modality !:

$$!A \quad = \quad \mathbf{Fam}(A) \quad = \quad \{ \quad (a_i)_{i \in I} \quad | \quad I \subseteq_{\mathrm{fin}} \mathbb{N}, \quad (a_i)_i \in A^I \quad \}$$

It induces a comonad on **Thin**, so that we get a **Kleisli bicategory Thin**$_!$ with

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ !A & & B \end{array}$$

as 1-cells.

Theorem (Clairambault, F.)

Thin$_!$ *is a cartesian closed bicategory.*

Other works

Other existing proof-relevant bicategorical models:

- **Generalized species of structures**
  Fiore, Gambino, et al. "The cartesian closed bicategory of generalised species of structures". 2008
- **Template games**
  Melliès. "Template games and differential linear logic". 2019

They rely on a **saturation** and/or **quotient** of witnesses w.r.t. symmetries.

In comparison, thin spans exhibit a **non-saturated** and **quotient-free** approach.

The end

Any questions?

Whiteboard