From Thin Concurrent Games to Generalized Species of Structures

Pierre Clairambault (Aix-Marseille Université)

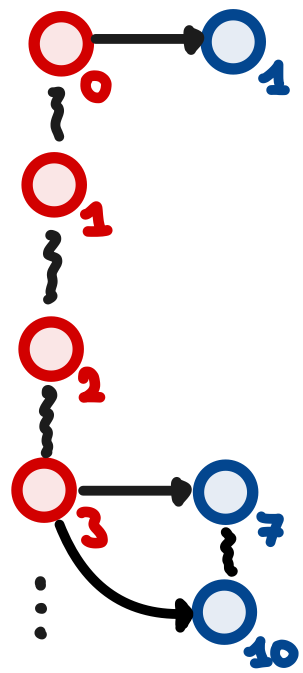
Federico Olimpieri (University of Leeds)

Hugo Paquet (LIPN, Université Sorbonne Paris Nord)

Two models

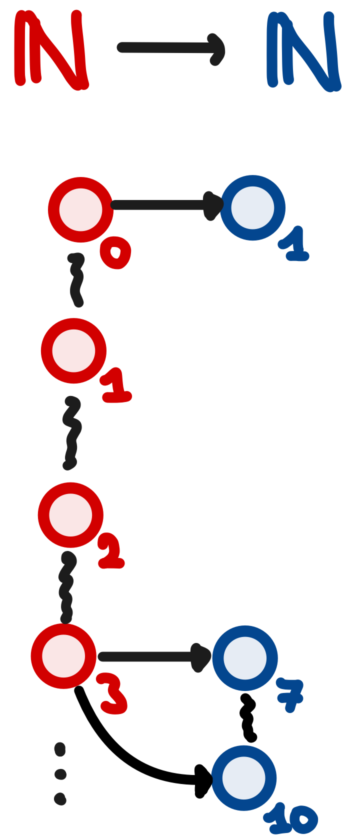
Game Semantics

$\mathbb{N} \longrightarrow \mathbb{N}$

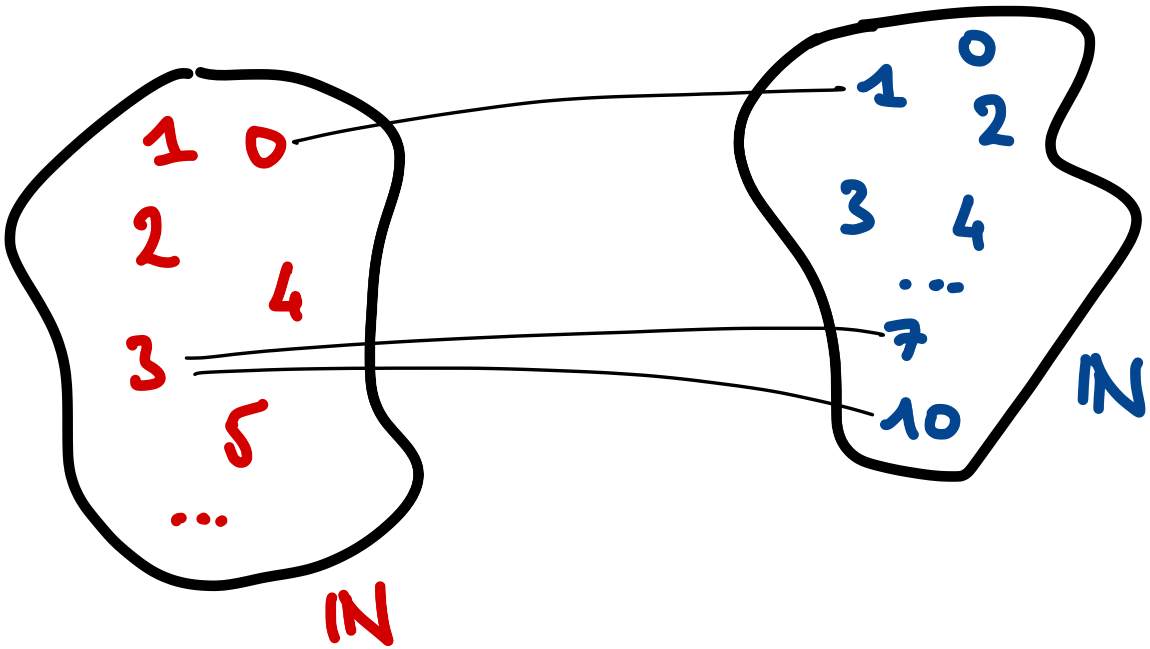


Two models

Game Semantics



Relational semantics



$$f, g \vdash \begin{array}{l} \text{let } n = f\,3 \\ \text{let } m = g\,4 \\ n + m \end{array}$$

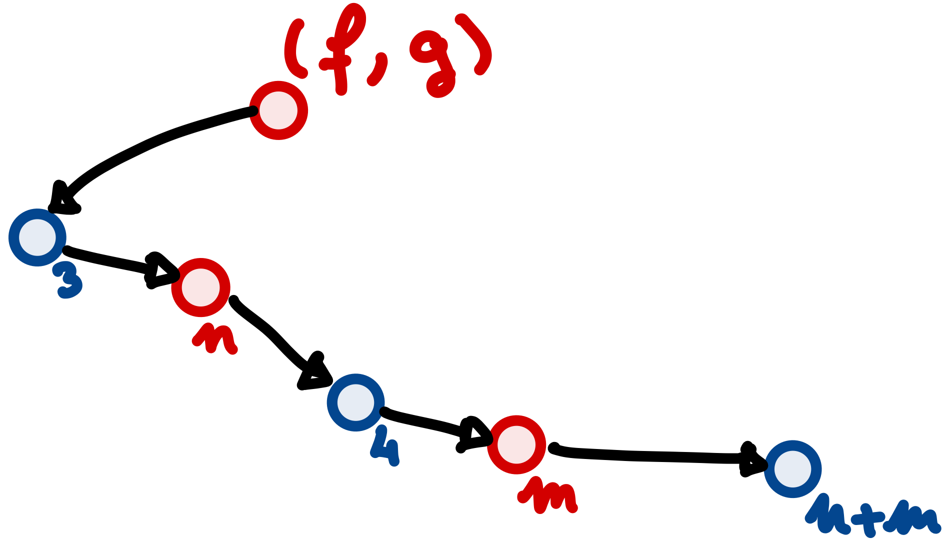
Game Semantics

Relational semantics

$$
\begin{aligned}
f, g \vdash\ & \mathrm{let}\ n = f\,3 \\
& \mathrm{let}\ m = g\,4 \\
& n + m
\end{aligned}
$$

Game Semantics

$$(\mathbb{N} \to \mathbb{N}) \times (\mathbb{N} \to \mathbb{N}) \longrightarrow \mathbb{N}$$

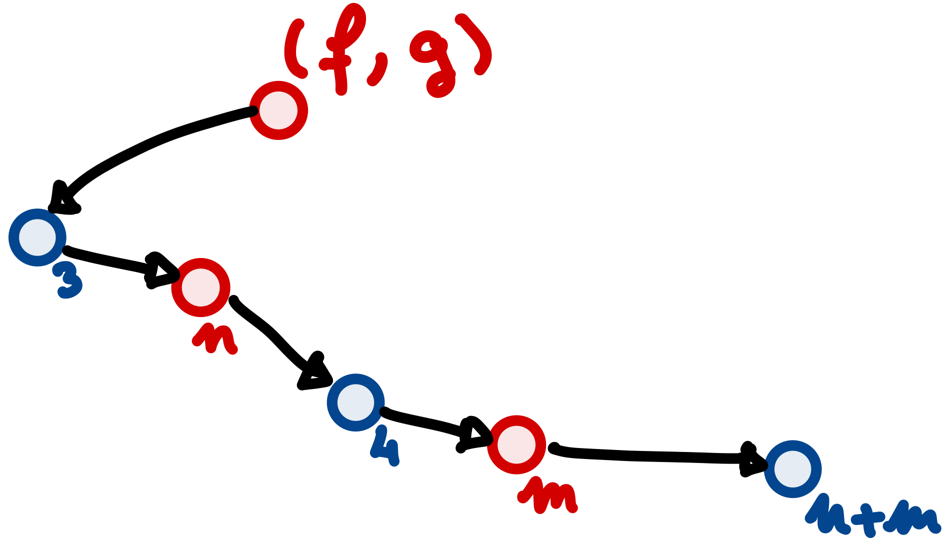


Relational semantics

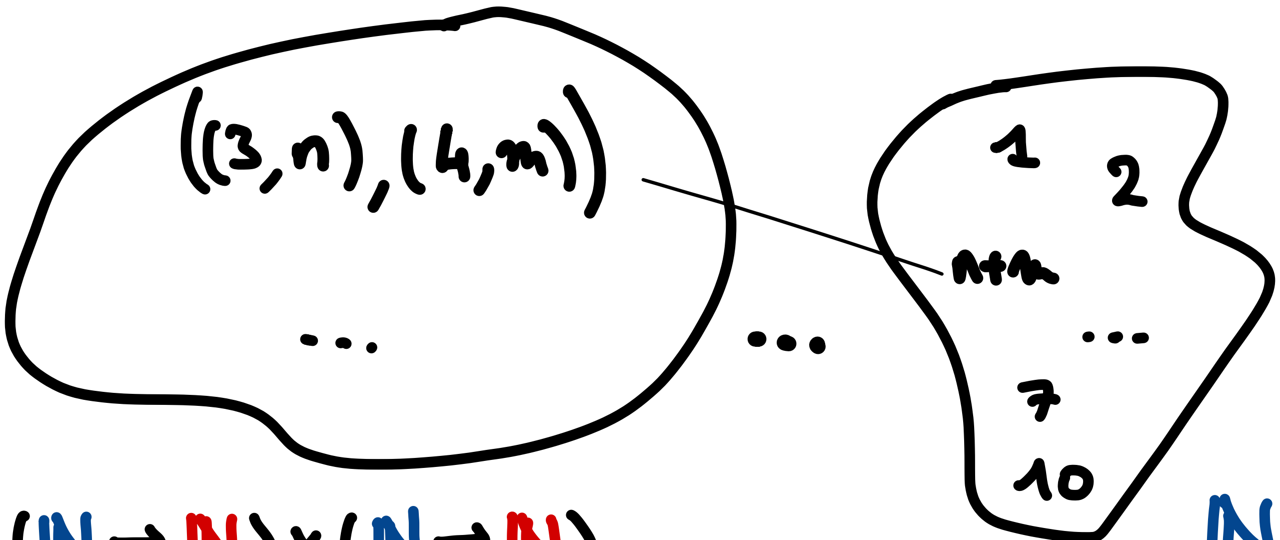
$$
f, g \vdash \begin{array}{l} \text{let } n = f\,3 \\ \text{let } m = g\,4 \\ n + m \end{array}
$$

Game Semantics

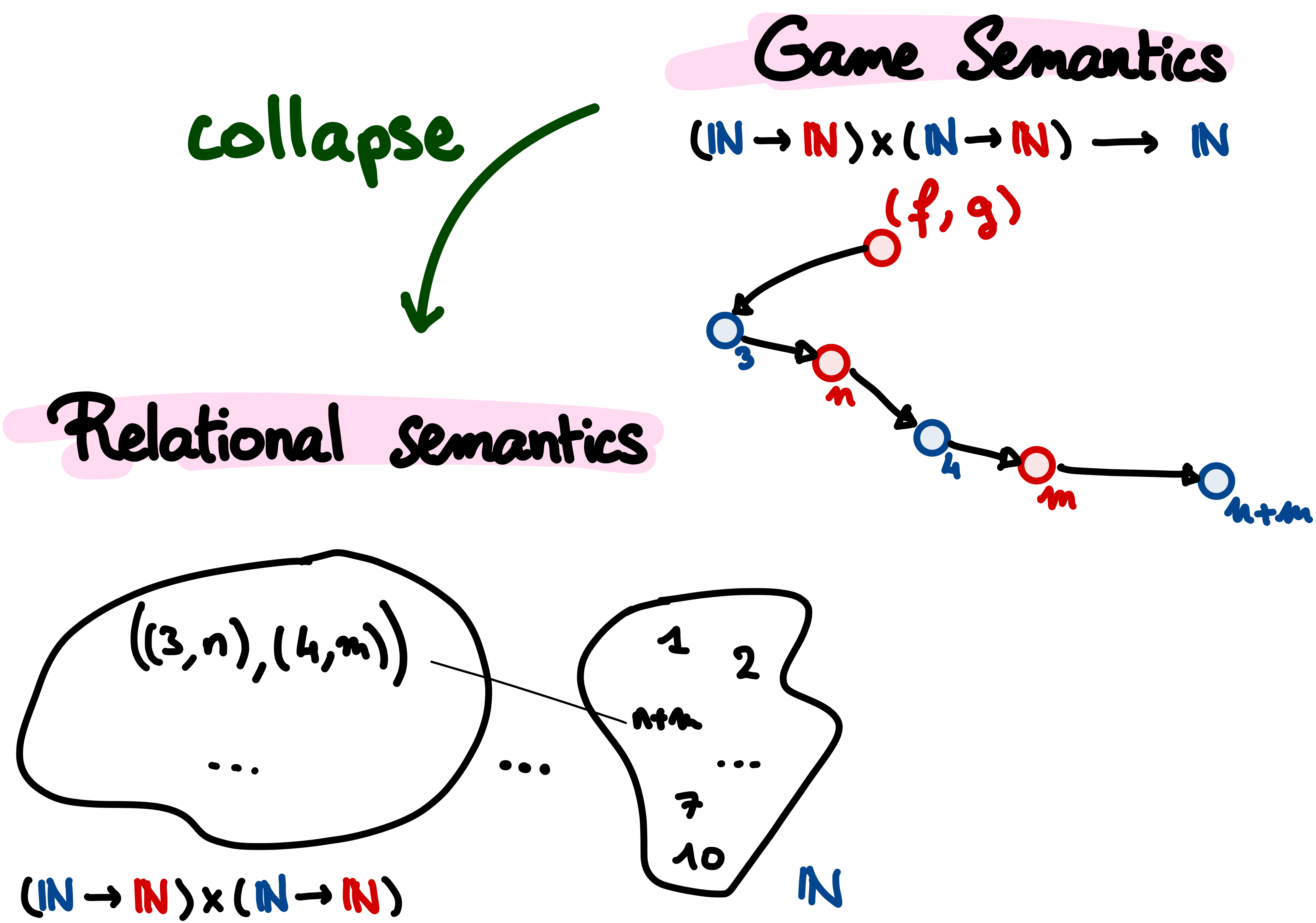
$(\mathbb{N} \to \mathbb{N}) \times (\mathbb{N} \to \mathbb{N}) \longrightarrow \mathbb{N}$

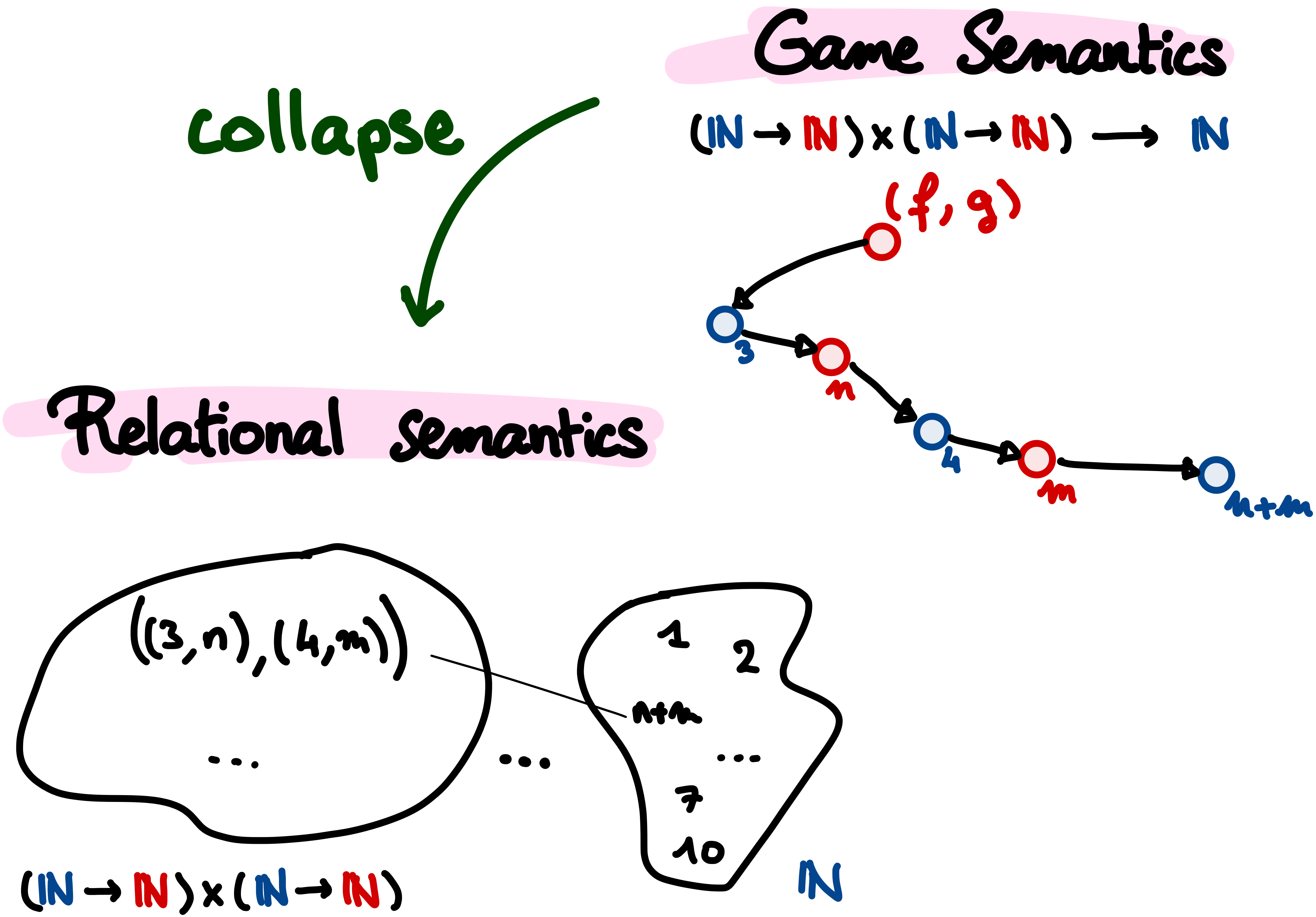


Relational semantics



$(\mathbb{N} \to \mathbb{N}) \times (\mathbb{N} \to \mathbb{N})$ $\mathbb{N}$


Game Semantics
(ℕ → ℕ) × (ℕ → ℕ) → ℕ
(f, g)
3
n
4
m
n+m
collapse
Relational semantics
((3,n),(4,m))
...
...
1
2
n+m
...
7
10
(ℕ → ℕ) × (ℕ → ℕ)
ℕ


Game Semantics
$(\mathbb{N} \to \mathbb{N}) \times (\mathbb{N} \to \mathbb{N}) \longrightarrow \mathbb{N}$
(f, g)
3
n
4
m
n+m
collapse
Relational semantics
((3, n), (4, m))
...
...
1
2
n+m
...
7
10
$(\mathbb{N} \to \mathbb{N}) \times (\mathbb{N} \to \mathbb{N})$
$\mathbb{N}$

Game Semantics

collapse ↓

Relational semantics

Background

- "Timeless games" (97)
- Polarization and tensorial logic (Melliès 00's)
- many others (Hyland, Shalk, McCusker, Winskel, ...)

This work

1. Proof-relevant models

(Relations ⟶ Profunctors)

2. Symmetry

Linear modality ! creates exchangeable copies

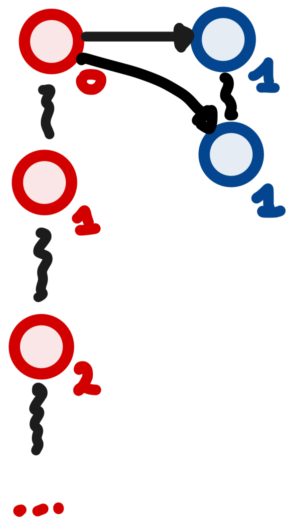
3. Bicategorical functors

$$\begin{tikzcd} \text{Games} \arrow[rr] & & \text{Prof} \\ & \text{Syntax} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof-relevant models

Game Semantics

$\mathbb{N} \longrightarrow \mathbb{N}$

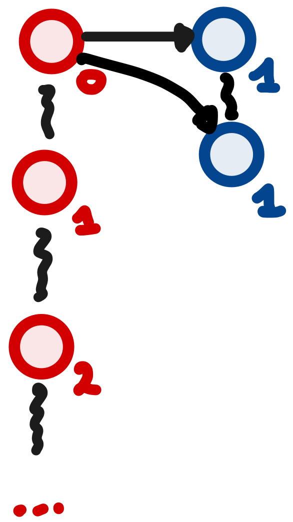


Relational semantics

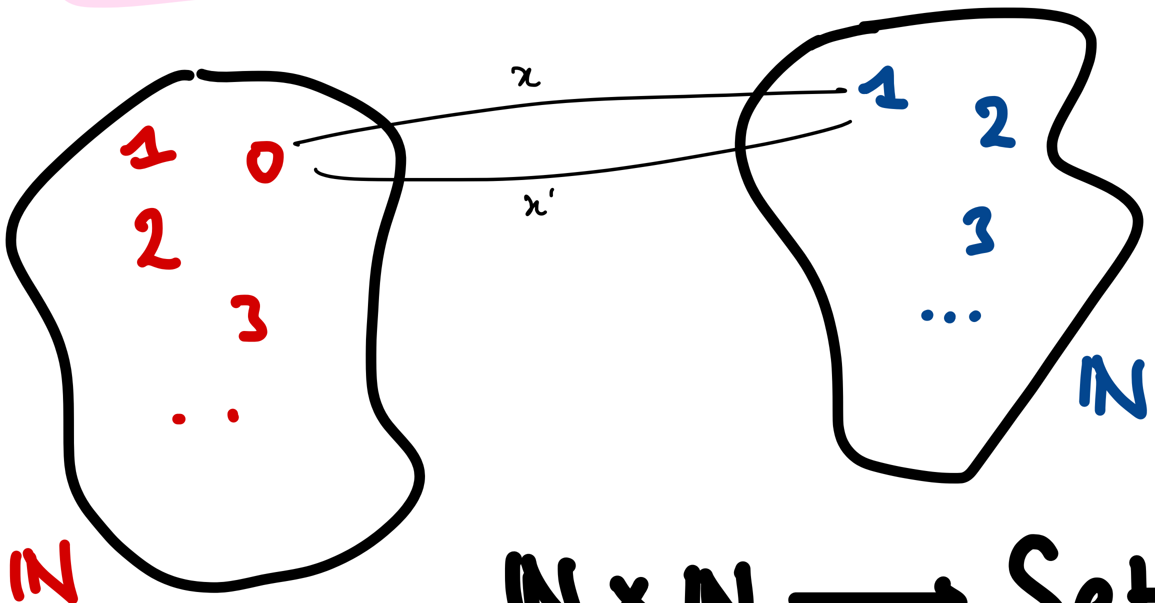
Proof-relevant models

Game Semantics

$\mathbb{N} \longrightarrow \mathbb{N}$



Relational semantics



$\mathbb{N} \times \mathbb{N} \longrightarrow \mathrm{Set}$

Strategy

$$\begin{tikzcd} \sigma \arrow[d] \\ A^{\perp} \parallel B \end{tikzcd}$$

$$\longrightarrow$$

Span of sets

$$\begin{tikzcd} & \mathcal{C}^{+}(\sigma) \arrow[dl] \arrow[dr] & \\ \mathcal{C}(A) & & \mathcal{C}(B) \end{tikzcd}$$

$$\longrightarrow$$

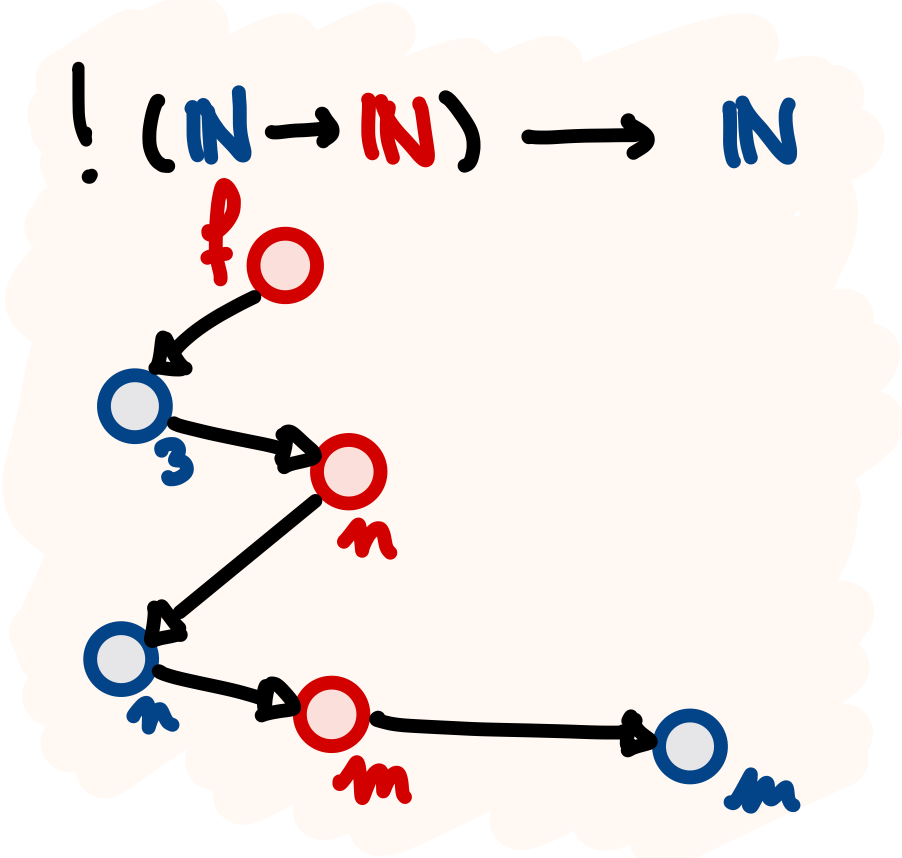
Profunctor

$$\begin{tikzcd} \mathcal{C}(A) \times \mathcal{C}(B) \arrow[d] \\ \mathrm{Set} \end{tikzcd}$$

Symmetry (Non-linear structure in proof-relevant models)

$$\lambda f . f\,(f\,3)$$

Symmetry (Non-linear structure in proof-relevant models)



$$\lambda f. f\,(f\,3)$$

Thin concurrent games

(Winskel 2007 + Castellan - Clairambault 2015)

$$!\,\circ \;=\; \circ_0 \quad \circ_1 \quad \circ_2 \quad \circ_3 \quad \circ_4 \quad \cdots$$

Symmetric copies

- Event structures with polarized symmetry:

  groupoids of configurations $\mathscr{C}^-(A),\ \mathscr{C}^+(A) \subseteq \mathscr{C}(A)$

- Uniformity condition on strategies

Generalized Species of Structures

(Fiore, Hyland, Gambino, Winskel 2008)

- Bicategory of profunctors between groupoids
- $!A$ is the free symmetric (strict) monoidal category
- $!A \times B^{op} \longrightarrow Set$ includes an action of the symmetry.

Theorem:

$$\begin{array}{c} \text{Thin concurrent games} \\ \big\downarrow \text{ freely generate the action} \\ \text{Generalized Species of Structures} \end{array}$$

A functor of bicategories

game A $\longmapsto$ groupoid $\mathcal{C}(A)$

strategy $A \overset{\sigma}{\nrightarrow} B$ $\longmapsto$ profunctor of witnesses

$\|\sigma\| : \mathcal{C}(A) \times \mathcal{C}(B)^{op} \longrightarrow \mathbf{Set}$

map of strategies

$$\begin{tikzcd} \sigma \arrow[rr, "f"] \arrow[dr] & & \sigma' \arrow[dl] \\ & A^{\perp} \otimes B & \end{tikzcd}$$

$\longmapsto$ natural transformation

$\|\sigma\| \longrightarrow \|\sigma'\|$

A functor of bicategories

game $A \longmapsto$ groupoid $\mathcal{C}(A)$

strategy $A \overset{\sigma}{\nrightarrow} B \longmapsto$ profunctor of witnesses

$\|\sigma\| : \mathcal{C}(A) \times \mathcal{C}(B)^{op} \longrightarrow \mathrm{Set}$

map of strategies

$$\begin{tikzcd} \sigma \arrow[rr, "f"] \arrow[dr] & & \sigma' \arrow[dl] \\ & A^{\perp} \otimes B & \end{tikzcd}$$

$\longmapsto$ natural transformation

$\|\sigma\| \longrightarrow \|\sigma'\|$

This is an oplax functor: $\|\tau \circ \sigma\| \underset{\neq}{\longrightarrow} \|\tau\| \circ \|\sigma\|$

A functor of bicategories

game A $\longmapsto$ groupoid $\mathscr{C}(A)$

strategy $A \xrightarrow{\sigma} B$ $\longmapsto$ profunctor of witnesses

$$\|\sigma\| : \mathscr{C}(A) \times \mathscr{C}(B)^{op} \longrightarrow \mathbf{Set}$$

map of strategies

$$\begin{tikzcd} \sigma \arrow[rr, "f"] \arrow[dr] & & \sigma' \arrow[dl] \\ & A^{\perp} \otimes B & \end{tikzcd} \quad \longmapsto \quad \text{natural transformation} \quad \|\sigma\| \longrightarrow \|\sigma'\|$$

pseudo if we restrict to "visible" strategies.

This is an ~~oplax~~ functor: $\|\tau \circ \sigma\| \cong \|\tau\| \circ \|\sigma\|$

Using additional structure on games (payoff)
we derive a cartesian closed pseudo-functor, so:

$$
\begin{tikzcd}
\text{Syntax} \arrow[r, "\llbracket - \rrbracket"] \arrow[d, "\llbracket - \rrbracket"'] & \text{Thin concurrent games} \arrow[dl] \\
\text{Generalized Species of Structures} &
\end{tikzcd}
$$

Summary

Two models with key differences
- composition mechanism
- representation of symmetry

We have made a formal connection in a proof-relevant setting.