Completeness for arbitrary finite dimensions of ZXW-calculus

arXiv:2302.12135

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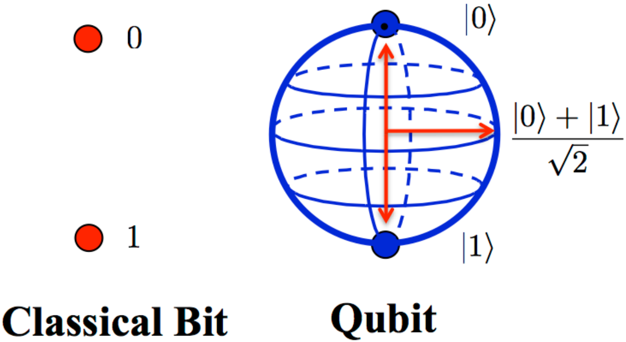
LICS '23

29th June 2023

QUANTINUUM

UNIVERSITY OF OXFORD

Bit vs qubit


0
1
Classical Bit
$|0\rangle$
$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$
$|1\rangle$
Qubit

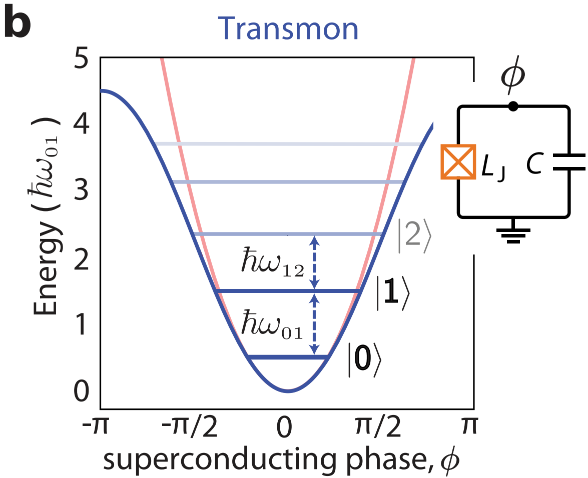
Qubits:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Qudits:

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + \cdots + a_{d-1}|d-1\rangle$$

Physical Realisation of Qudits



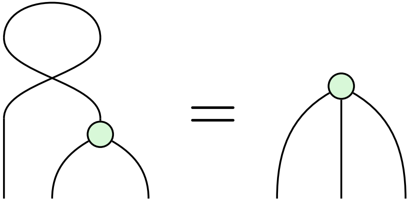
Kjaergaard et al., 2020

Benefits of qudits

- Native simulation of certain systems
- Improved quantum error correction
- Simplified quantum algorithms

Graphical language for quantum computing

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}^T$$



Completeness

A graphical calculus is complete if we can derive $D_1 = D_2$ from the rules of the calculus, given that the interpretation of D_1 and D_2 equal.

Qubit ZX-calculus

- Formulated by Coecke and Duncan, 2007
- Universal completeness (Ng and Wang, 2017; Jeandel, Perdrix and Vilmart, 2018)

Qubit ZX-calculus

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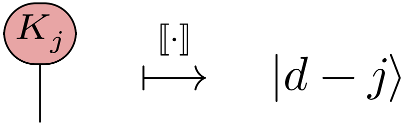
- Universal completeness (Ng and Wang, 2017; Jeandel, Perdrix and Vilmart, 2018)
- Circuit simplification
- Error-correction

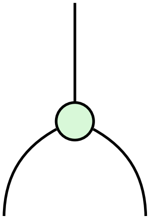
The qudit ZXW-calculus

Standard basis in qudit ZXW

For $0 \leq j < d$,

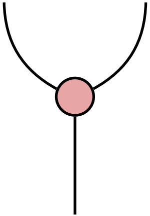


Z spider



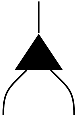
$$\overset{[\![\cdot]\!]}{\longmapsto} \quad |k\rangle \mapsto |k,\ k\rangle$$

X spider



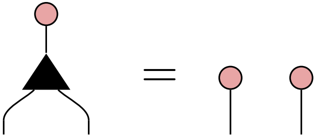
$$\xrightarrow{[\![\cdot]\!]} \quad |i,\ j\rangle \mapsto |i \oplus_d j\rangle$$

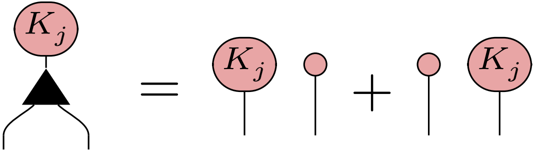
Generator: W node



$$\overset{[\![\cdot]\!]}{\longmapsto} \quad |00\rangle\langle 0| + \sum_{i=1}^{d-1}(|i0\rangle + |0i\rangle)\langle i|$$

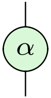
That is:





Understanding the Z box

Z spider:



$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}, \qquad \text{where } \alpha \in \mathbb{R}.$$

Z box:

$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \qquad \text{where } a \in \mathbb{C}.$$

Understanding the qudit Z box

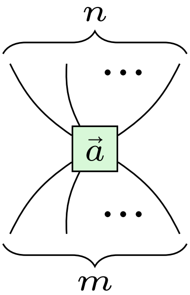
Qubit Z box: for $a \in \mathbb{C}$,

$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

Qudit Z box: for $\vec{a} = (a_1, a_2, \cdots, a_{d-1}) \in \mathbb{C}^{d-1}$,

$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{d-1} \end{bmatrix}$$

Generator: Z box

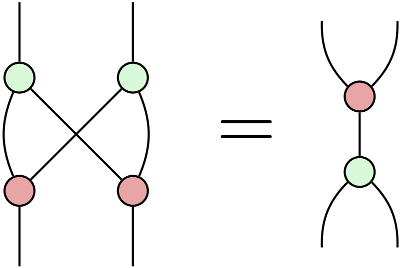


$$\overset{[\![\cdot]\!]}{\longmapsto} \quad \sum_{j=0}^{d-1} a_j \, |j\rangle^{\otimes m} \, \langle j|^{\otimes n} \,,$$

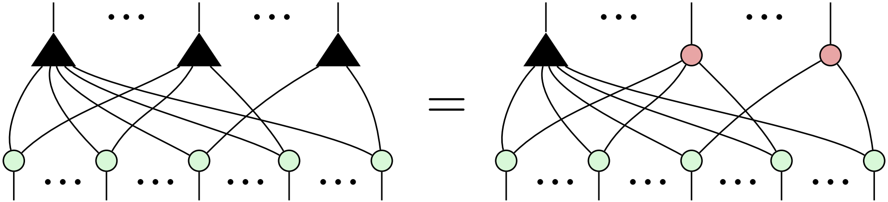
where $\vec{a} = (a_1, \cdots, a_{d-1}) \in \mathbb{C}$

and $a_0 := 1.$

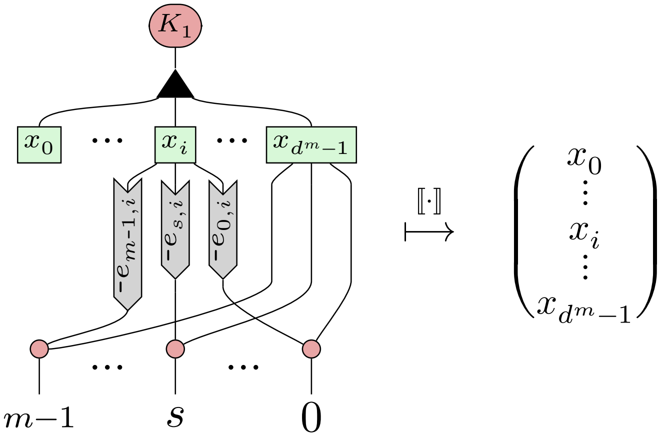
Rule: Bialgebra



Rule: Trialgebra

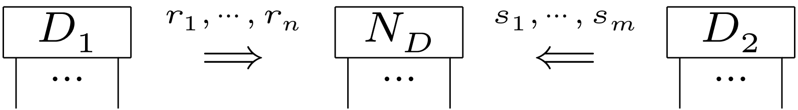


A Normal Form



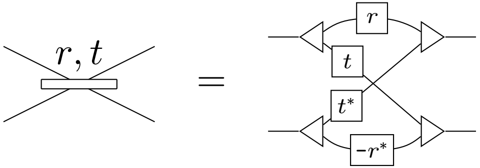
Completeness using a normal form

If $[\![D_1]\!] = [\![D_2]\!]$, then:

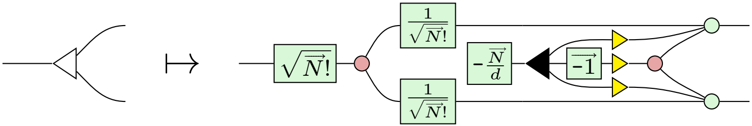


Application: Photonics

A beamsplitter:

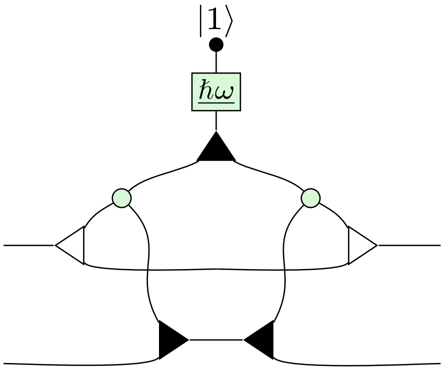


where



Application: Hamiltonians

$$H_{JC} = \hbar\omega \left(a_1 \sigma_2^+ + a_1^\dagger \sigma_2^- \right)$$

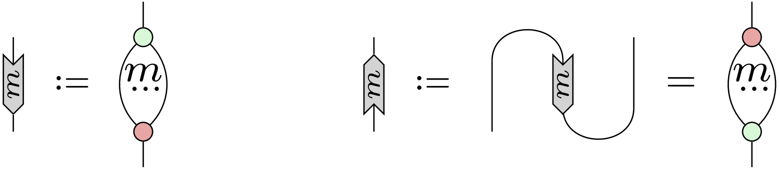


Outlook

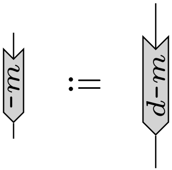
- Simplification of photonics circuits using ZXW
- Hamiltionian simplification with ZXW
- Completeness of qufinite ZXW-calculus

Appendix

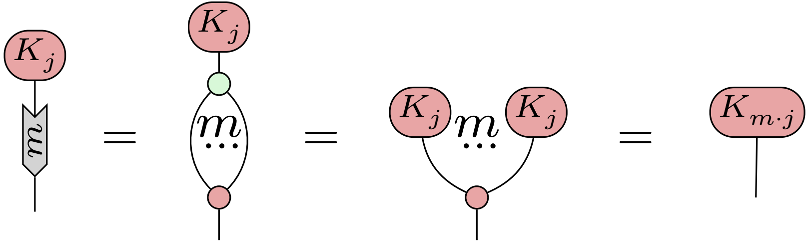
Notation: The multiplier



m can be labeled modulo d due to the Hopf law.



Example



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References II

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