Titouan Carette, University of Latvia
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Complete Graphical Language For Hermiticity-Preserving Superoperators

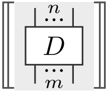
June 29, 2023

LICS 2023

Chapter 1

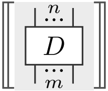
Picturing processes

Diagrams



$\in \mathcal{M}_{2^m \times 2^n}$

Diagrams



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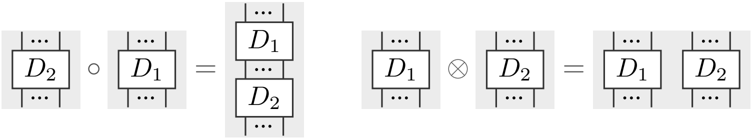

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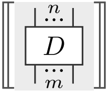

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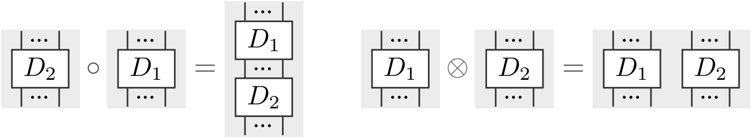

- Completeness:

$$[\![A]\!] = [\![B]\!] \Leftrightarrow A = B \text{ modulo rewriting rules.}$$

Completeness Fashion

ZX-Calculus **ZW-Calculus** ZH-Calculus

- **A Complete Equational Theory for Quantum Circuits**, *LICS*, Alexandre Clément, Nicolas Heurtel, Shane Mansfield, Simon Perdrix and Benoît Valiron

- **Completeness for arbitrary finite dimensions of ZXW-calculus, a unifying calculus**, *LICS2023*, Boldizsár Poór, Quanlong Wang, Razin Shaikh, Lia Yeh, Richie Yeung and Bob Coecke

Chapter 2

Anti-unitaries

Time reversal reflection on the Bloch sphere

$$\{(a, b) \in \mathbb{C}^2, |a|^2 + |b|^2 = 1\}$$

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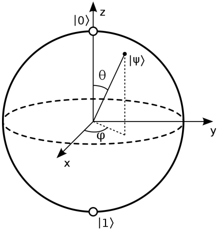

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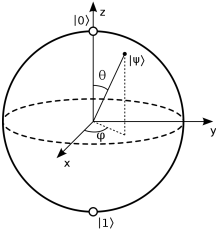

Unitaries

Time reversal reflection on the Bloch sphere

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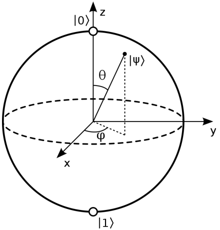
Rotations

$$O_3 = SO_3 \rtimes \mathbb{Z}_2$$



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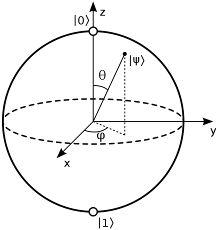


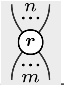
Unitaries

$$U_2 \rtimes \mathbb{Z}_2$$

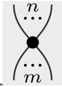
- Anti-unitary:

$$K : (a, b) \mapsto (\overline{a}, \overline{b}) \qquad K^2 = I \qquad KU_t = Ke^{itH} = e^{-itH^t} K$$

The ZW-Calculus

⟦ 

⟧ $= |0^m\rangle\langle 0^n| + r\,|1^m\rangle\langle 1^n|$

⟦ 

⟧ $= \sum_{\substack{x,y\in\{0,1\}^{n+m} \\ |x\cdot y|=1}} |y\rangle\langle x|$

⟦  ⟧ $= \sum_{i,j\in\{0,1\}} (-1)^{ij}\,|ji\rangle\langle ij|$

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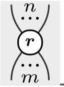
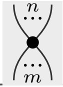
⟦  ⟧ = ⟦  ⟧$^\dagger = |00\rangle + |11\rangle$

The ZW-Calculus

$[\![$  $]\!] = |0^m\rangle\langle 0^n| + r\,|1^m\rangle\langle 1^n|$

$[\![$  $]\!] = \sum_{\substack{x,y \in \{0,1\}^{n+m} \\ |x \cdot y| = 1}} |y\rangle\langle x|$

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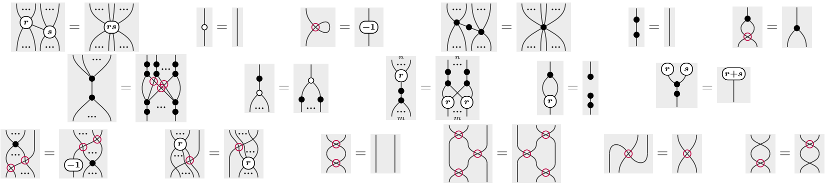
Axioms:



A first attempt

 $: 1 \to 1$ $\qquad$ $\Big[\!\!\Big[$  $\Big]\!\!\Big] = K$

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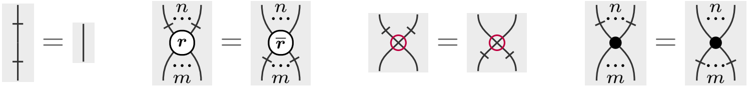

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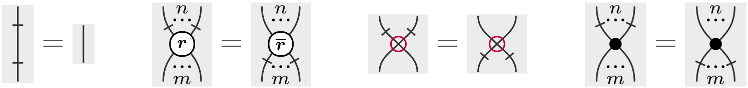

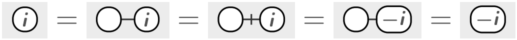

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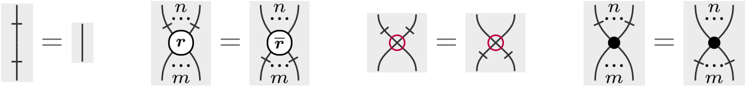
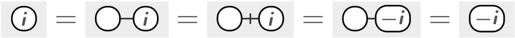
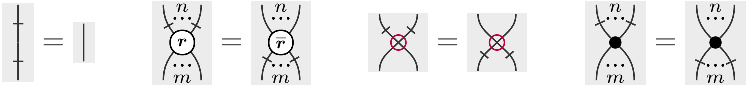
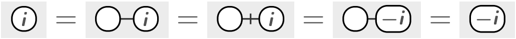




- But ⟦⟧ $= 1 + i \neq 1 - i =$ ⟦⟧, a contradiction!

A first attempt



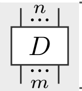


- But $[\![\text{(i)}]\!] = 1 + i \neq 1 - i = [\![\text{(−i)}]\!]$, a contradiction!
- K is $\mathbb{R}$-linear, but **not** $\mathbb{C}$-linear!

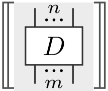
Chapter 3

Superoperators

Doubling And Unzipping

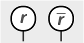
$\llbracket$  $\rrbracket : \mathcal{M}_{2^n \times 2^n}(\mathbb{C}) \to \mathcal{M}_{2^m \times 2^m}(\mathbb{C})$

Doubling And Unzipping



$$: \mathcal{M}_{2^n \times 2^n}(\mathbb{C}) \to \mathcal{M}_{2^m \times 2^m}(\mathbb{C})$$

- Density matrix: $x \in \mathbb{C}^{2^n} \leftrightarrow xx^\dagger \in \mathcal{M}_{2^n \times 2^n}(\mathbb{C})$.

unzip () = 

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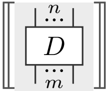

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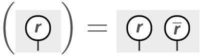
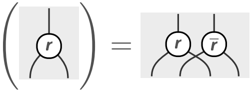
unzip 

- Pure maps: $U \in \mathcal{M}_{2^m \times 2^n}(\mathbb{C}) \leftrightarrow \rho \mapsto U\rho U^\dagger$.

unzip 

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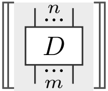

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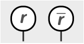
unzip (

) = 

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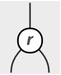
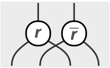
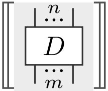
unzip (

) = 

- Discard: $\rho \mapsto \mathrm{Tr}(\rho)$. unzip () = 

Doubling And Unzipping



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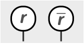
unzip () = 

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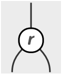
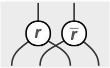
unzip () = 

- Discard: $\rho \mapsto \mathrm{Tr}(\rho)$. unzip () = 

- Tick: $x \mapsto \overline{x} \leftrightarrow \rho \mapsto \rho^t$. unzip() = .

Positivity Versus Hermitianity

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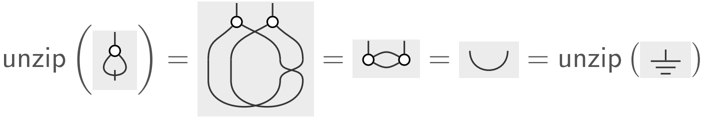
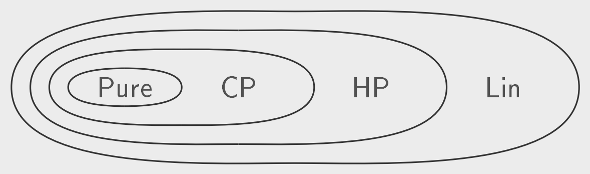


Quantum Process Theory Hierarchy



Prop	Processes	States
Pure	pure	rank-1 positive Hermitian matrices
CP	CP	positive Hermitian matrices
HP	HP	Hermitian matrices
Lin	linear	matrices

Chapter 4

Completeness

The lesson of Completely Positive maps

- **Quantum channels as a categorical completion**, *LICS2019*, Mathieu Huot, Sam Staton.

- **Completeness of Graphical Languages for Mixed State Quantum Mechanics**, *ACM Transactions of Quantum Computing*, Titouan Carette, Simon Perdrix, Renaud Vilmart, Emmanuel Jeandel.

The recipe :

1. Characterize the extension as a universal construction
2. Translate the universal property in equations
3. Lift the completeness of your favorite language

Blinded by Obsession

The quest for essential uniqueness of purification.

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- Uniqueness up to isometries.
- Related to singular value decomposition.
- Very well studied, lots of results.

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- Almost nothing.

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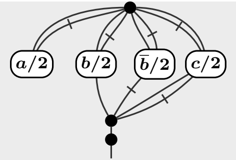
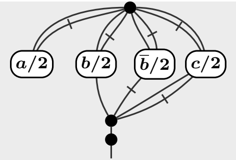
Hermitianity is **weaker** but **simpler** than positivity!

Bruteforcing completeness



$$\rightarrow \begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix}$$

Bruteforcing completeness



$$\to \begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix}$$