A Complete Equational Theory for Quantum Circuits

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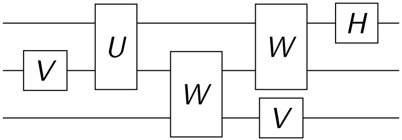

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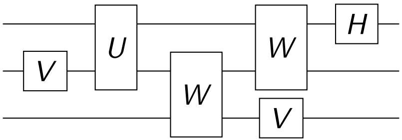

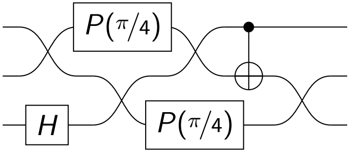

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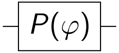

phase gate
($\simeq$ Z-rotation)

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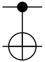

controlled
not

Structure of PROP:

- Additional generators:

empty
circuit

identity

swap

- The circuits are built by means of:

sequential
composition
$C_2 \circ C_1$

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- Deformation rules, e.g.

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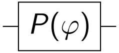

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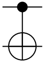

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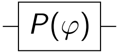

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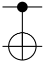

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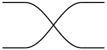

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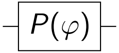

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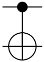

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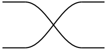

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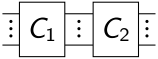

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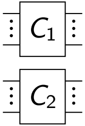

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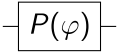

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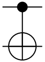

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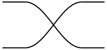

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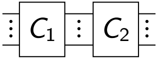

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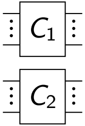

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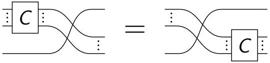
- Deformation rules, e.g. 

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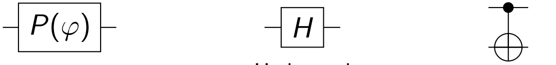

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We define:

$Z := P(\pi)$ $\quad$ $X := H\,Z\,H$ $\quad$ $R_X(\theta) := H\,P(\theta)\,H$

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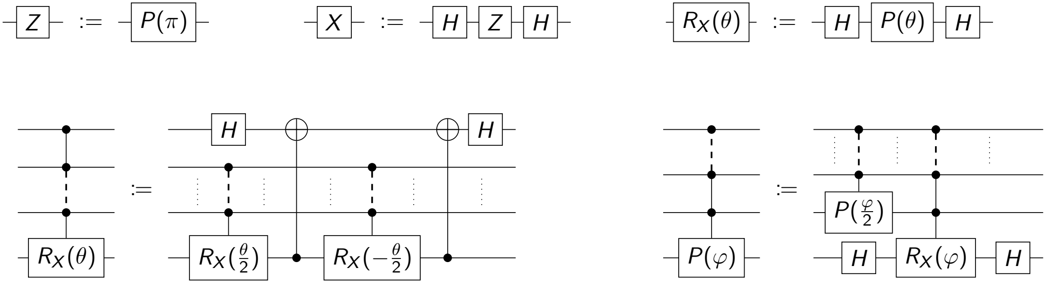

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Equational Theories for Quantum Circuits

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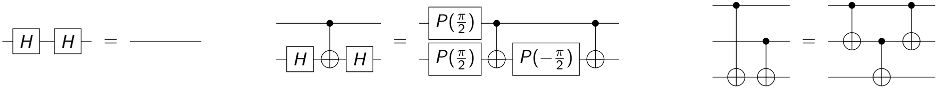

- Common tool for doing circuit transformations
- Used in particular for circuit optimisation

It is complete if any two circuits representing the same unitary can be transformed into each other.

- Provides theoretical foundations e.g. for defining rewriting strategies
- Open problem for 30 years

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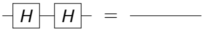

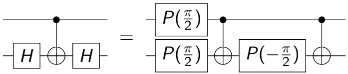

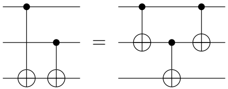

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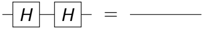

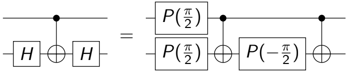

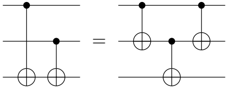

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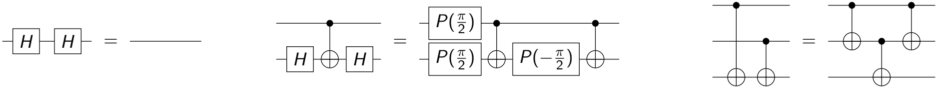

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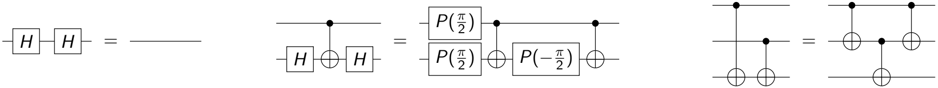

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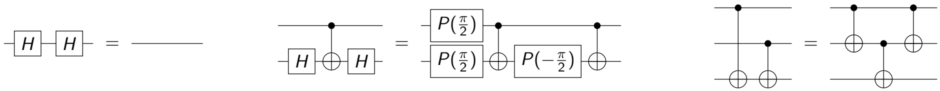

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Partial Results

- Complete equational theories for non-universal fragments, e.g.:
  - circuits on at most 2 qubits [X. Bian, P. Selinger, '18]
  - stabilizer circuits [A. Ranchin, B. Coecke, '14, J. Makary, N. J. Ross, P. Selinger, '21]
  - CNot-dihedral circuits [M. Amy, J. Chen, N. J. Ross, '18]
- ZX-calculus: generalisation of quantum circuits
  - more expressive than quantum circuits (can represent all matrices in $\mathbb{C}^{m\times n}$)
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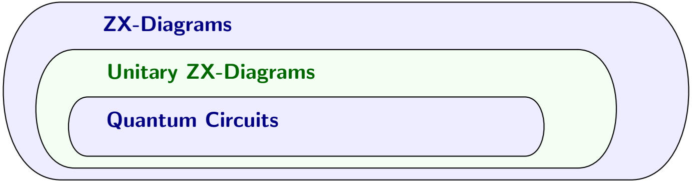

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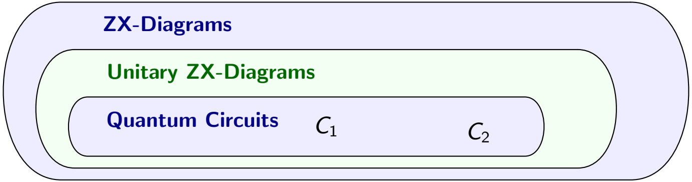

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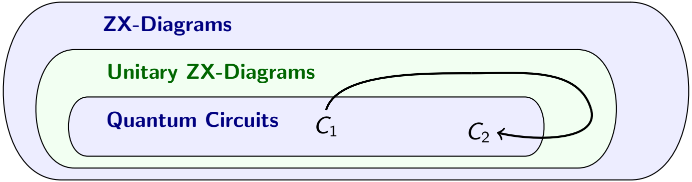
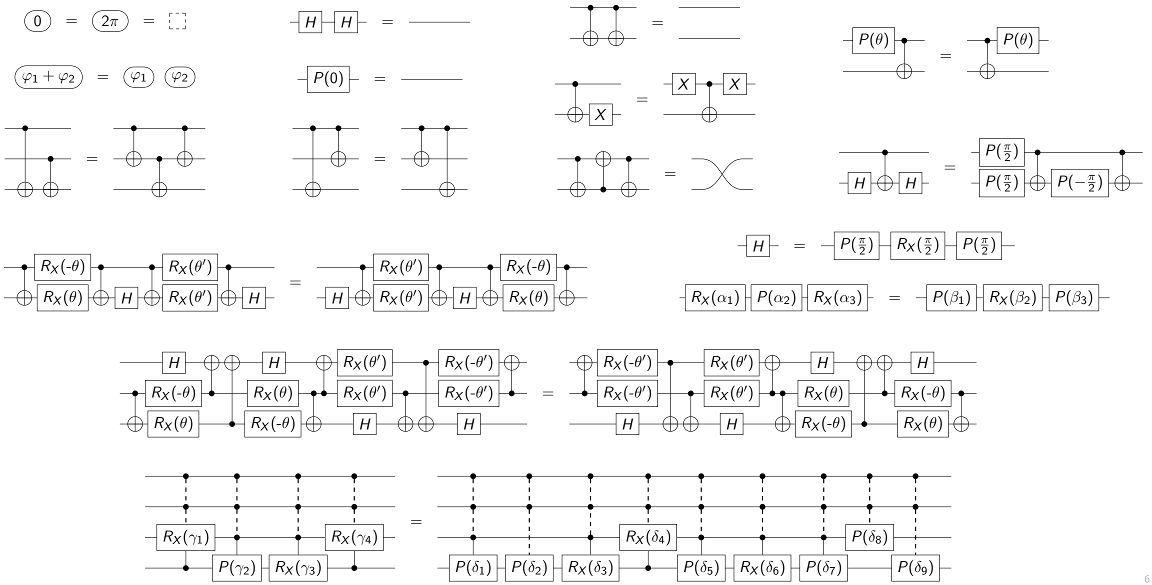


A Complete Equational Theory for Quantum Circuits



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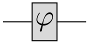
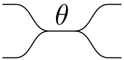


phase shifter



beam splitter

together with:

by means of:

sequential composition

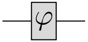
$D_2 \circ D_1$

parallel composition

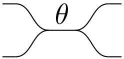
$D_1 \oplus D_2$

Structure of PROP

LOPP-circuits are generated by:

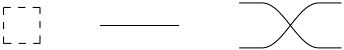


phase shifter

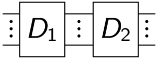


beam splitter

together with:

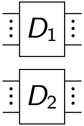


by means of:



sequential composition

$D_2 \circ D_1$



parallel composition

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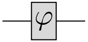
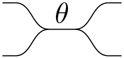


phase shifter

$e^{i\varphi}$



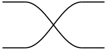
beam splitter

$$\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

together with:

$()$

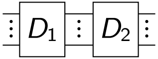
1



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Structure of PROP

by means of:



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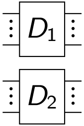

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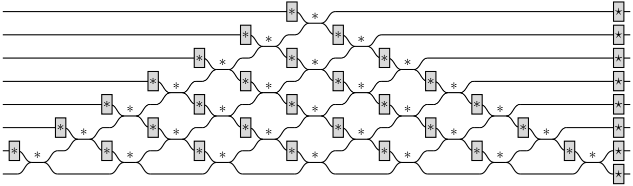
$$\left(\begin{array}{c|c} [\![D_1]\!] & 0 \\ \hline 0 & [\![D_2]\!] \end{array}\right)$$

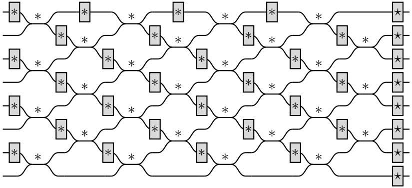
Universality of LOPP-Circuits

Proposition

For any unitary $U \in \mathbb{C}^{n\times n}$, there exists a LOPP-circuit D such that $[\![D]\!] = U$.

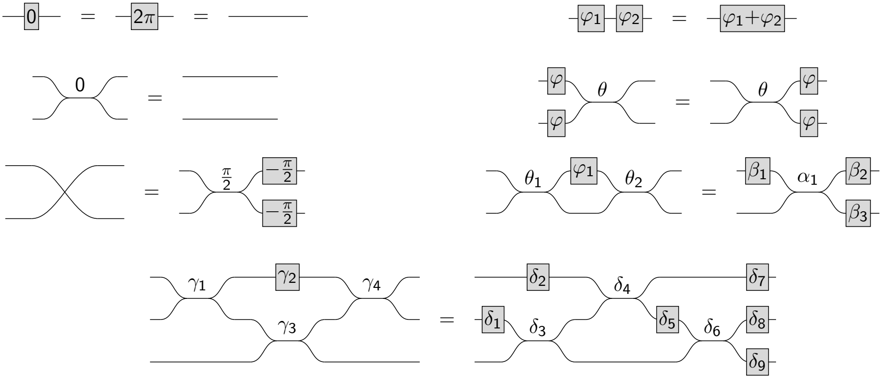


Reck *et al.* (1994)



Clements *et al.* (2016)

Complete Equational Theory for LOPP-Circuits



Proposition (Soundness)

$\forall D_1, D_2$, if $\text{LOPP} \vdash D_1 = D_2$ then $[\![D_1]\!] = [\![D_2]\!]$.

Theorem (Completeness)

$\forall D_1, D_2$, *if* $[\![D_1]\!] = [\![D_2]\!]$ *then* $\text{LOPP} \vdash D_1 = D_2$.

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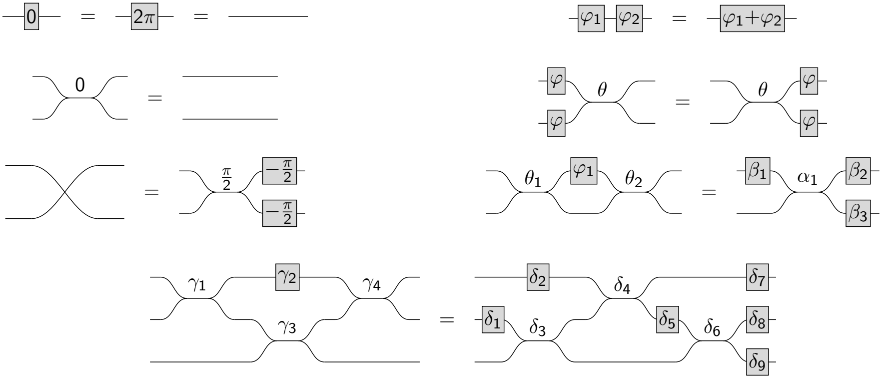

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n-Qubit Quantum Circuits

C_1 C_2

2^n-Mode Optical Circuits

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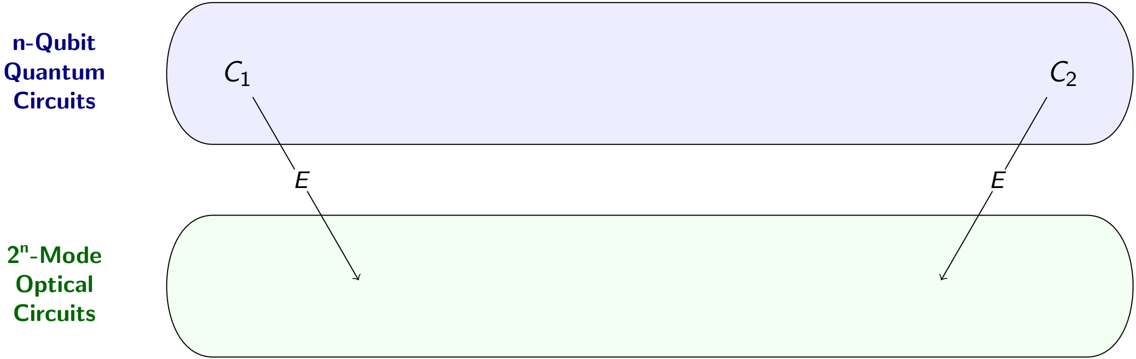

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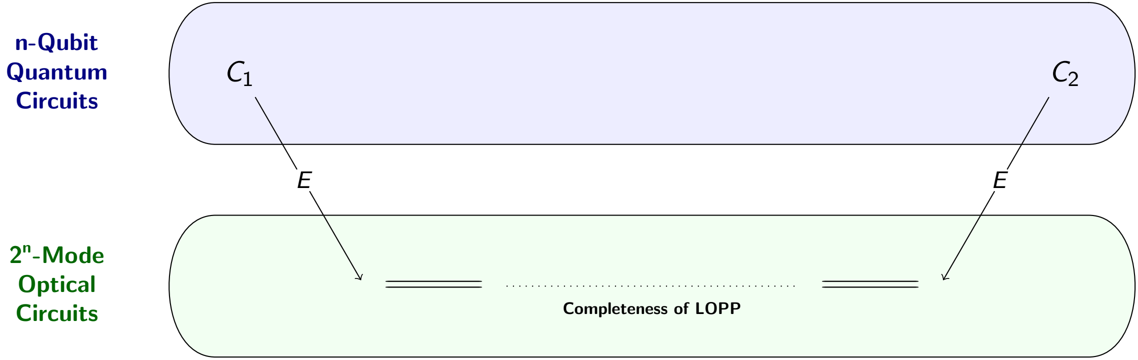

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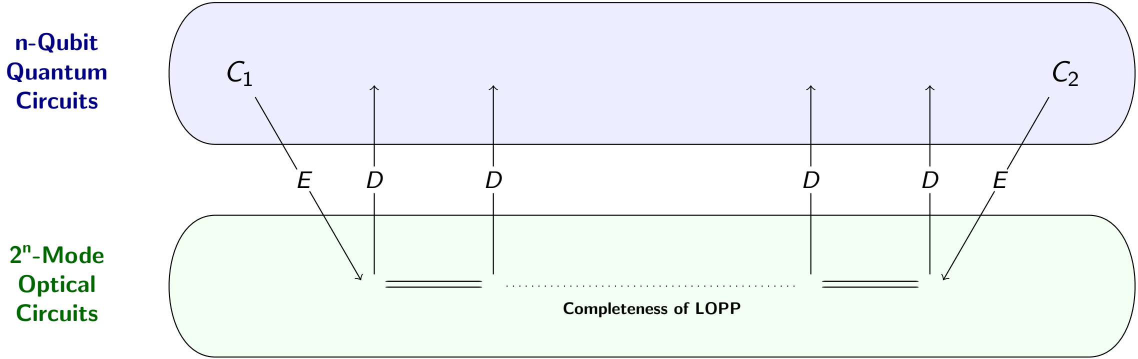

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Lemma 1

For any quantum circuit C, $\mathrm{QC} \vdash D(E(C)) = C$.

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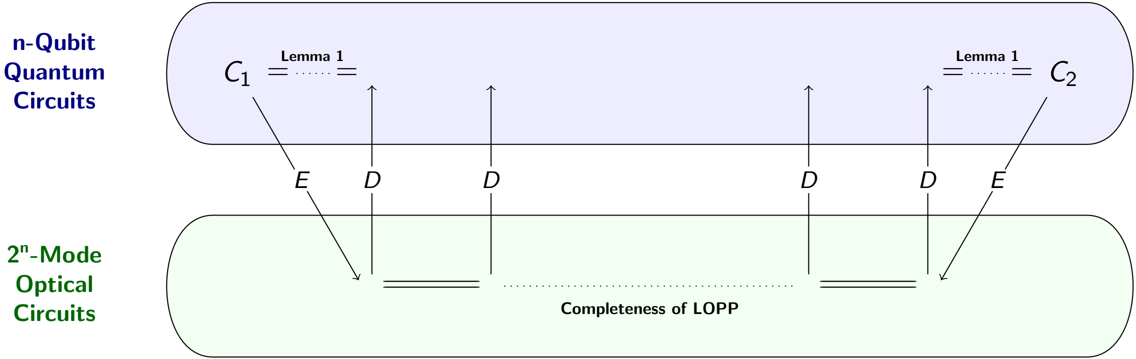
For every equation of the LOPP-calculus, of the form $D_1 = D_2$, *one has* $\mathrm{QC} \vdash D(D_1) = D(D_2)$.



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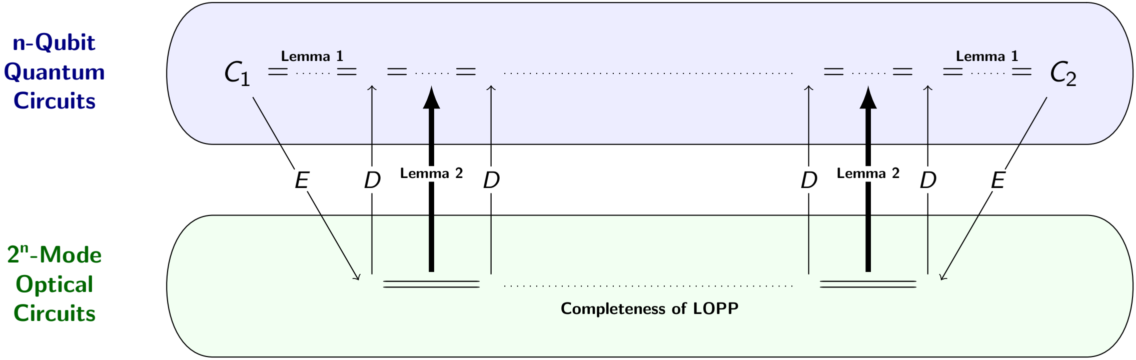
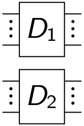


Key Aspects of the Proof

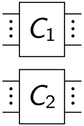
- Note that optical circuits and quantum circuits have different structures:

Optical circuits:



$D_1 \oplus D_2$

Quantum circuits:



$C_1 \otimes C_2$

$\Rightarrow$ To define E and D, we "sequentialise" the circuits:

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- Decoding produces multi-controlled gates, e.g.:

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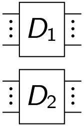

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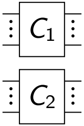

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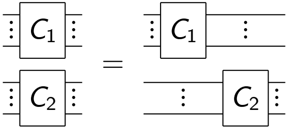

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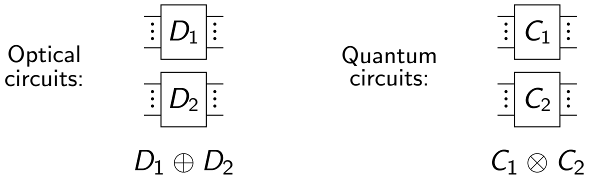

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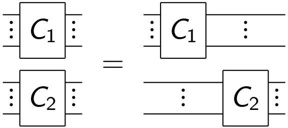

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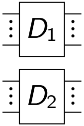

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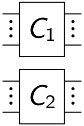

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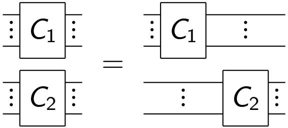

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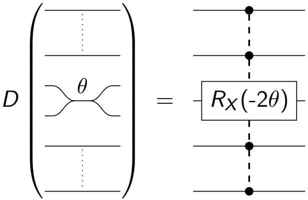

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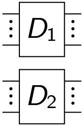

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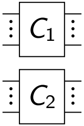

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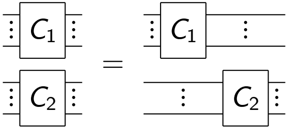

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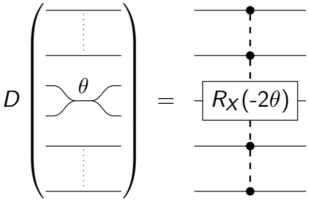

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \cos\theta & i\sin\theta & & \\ & & & i\sin\theta & \cos\theta & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix}$$

two-level matrix

Future Work

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- Design procedures e.g. for circuit optimisation
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