A Complete Equational Theory for Quantum Circuits

Alexandre Clément*, Nicolas Heurtel†‡, Shane Mansfield†, Simon Perdrix*, Benoît Valiron‡¶

*Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France
†Quandela, 7 rue Léonard de Vinci, 91300 Massy, France
‡Université Paris-Saclay, Inria, CNRS, ENS Paris-Saclay, LMF, 91190, Gif-sur-Yvette, France
¶CentraleSupélec, 91190, Gif-sur-Yvette, France

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Quantum Circuits

- Roughly speaking, they are the assembly language of a quantum processor.
- Therefore ubiquitous in quantum computing.
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A Complete Equational Theory for Quantum Circuits
We consider the PROP of quantum circuits generated by:

\[
\begin{align*}
P(\varphi) & \quad \text{phase gate} \\
H & \quad \text{Hadamard} \\
\end{align*}
\]

\[
\begin{align*}
\sim \text{Z-rotation} \\
\text{controlled not} \\
\end{align*}
\]

**Structure of PROP:**
- **Additional generators:**
  - empty circuit
  - identity
  - swap

- **The circuits are built by means of:**
  - sequential composition \( C_2 \circ C_1 \)
  - parallel composition \( C_1 \otimes C_2 \)

**Deformation rules, e.g.**

\[
\begin{align*}
\text{C} \quad \text{C} &= \quad \text{C} \quad \text{C} \\
\end{align*}
\]
Quantum Circuits

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We consider the PROP of quantum circuits generated by:

- $P(\varphi)$: phase gate ($\approx Z$-rotation)
- $H$: Hadamard
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  - sequential composition $C_2 \circ C_1$
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Deformation rules, e.g.

$C \odot C = C \otimes C$
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\[ P(\varphi) \]

- phase gate 
  \( \approx \text{Z-rotation} \)

\[ H \]

- Hadamard

\[ \text{controlled not} \]

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  - sequential composition
    \( C_2 \circ C_1 \)
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- Deformation rules, e.g.
  \[ C \circ (C_2 \circ C_1) = (C \circ C_2) \circ C_1 \]
Quantum Circuits

We consider the PROP of quantum circuits generated by:

- $P(\varphi)$: phase gate ($\simeq Z$-rotation)
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We define:

- $Z := P(\pi)$
- $X := HZH$
- $R_X(\theta) := H P(\theta) H$
- $R_X(\frac{\theta}{2})$
- $R_X(-\frac{\theta}{2})$
- $P(\varphi)$

We consider the PROP of quantum circuits generated by:

- \( P(\phi) \) phase gate (\( \approx \) Z-rotation)
- \( H \) Hadamard
- controlled not

We define:

\[
Z := P(\pi) \\
X := HZH \\
R_X(\theta) := H P(\theta) H \\
R_X(-\theta/2) := H \\
P(\varphi) := H R_X(\varphi) H
\]

Quantum Circuits

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- $R_X(\theta) := H P(\theta) H$

An equational theory is a set of equalities between circuits, e.g.:

\[
\begin{align*}
\begin{array}{cccc}
H & H & = & \quad \quad \quad \quad \quad \\
& & \quad & \\
& & \quad & \\
H & H & = & \quad \quad \quad \quad \quad \\
& & P(\frac{\pi}{2}) & \\
& & P(-\frac{\pi}{2}) & \\
& & & \\
\end{array}
\end{align*}
\]

- Common tool for doing circuit transformations
- Used in particular for circuit optimisation

It is complete if any two circuits representing the same unitary can be transformed into each other.

- Provides theoretical foundations e.g. for defining rewriting strategies
- Open problem for 30 years
An equational theory is a set of equalities between circuits, e.g.:

- $H \cdot H = H$
- $H \oplus H = P\left(\frac{\pi}{2}\right) \cdot P\left(-\frac{\pi}{2}\right)$
- $P\left(\frac{\pi}{2}\right) \cdot P\left(-\frac{\pi}{2}\right) = 1$

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- Used in particular for circuit optimisation

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- $H \, H = H$
- $H \oplus H = P(\frac{\pi}{2}) \oplus P(-\frac{\pi}{2})$
- $P(\frac{\pi}{2}) \, P(\frac{\pi}{2}) = P(\pi) \, P(-\pi)$

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\[ \begin{align*}
    H \otimes H &= \quad \\
    H \otimes H &= P(\frac{\pi}{2}) \otimes P(-\frac{\pi}{2})
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Partial Results

- Complete equational theories for non-universal fragments, e.g.:
  - circuits on at most 2 qubits [X. Bian, P. Selinger, ’18]
  - CNot-dihedral circuits [M. Amy, J. Chen, N. J. Ross, ’18]

- ZX-calculus: generalisation of quantum circuits
  - more expressive than quantum circuits (can represent all matrices in $\mathbb{C}^{m \times n}$)
  - equipped with a complete equational theory [E. Jeandel, S. Perdrix, R. Vilmart, ’18]
  - extracting a quantum circuit from a unitary ZX-diagram is $\#P$-hard in general
  - completeness of ZX-calculus does not lead to completeness inside quantum circuits.
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Unitary ZX-Diagrams

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LOPP-circuits are generated by:

- phase shifter
- beam splitter

Structure of PROP

by means of:

- sequential composition: $D_2 \circ D_1$
- parallel composition: $D_1 \oplus D_2$
LOPP-circuits are generated by:

\[
\begin{align*}
\phi & \quad \text{phase shifter} \\
\theta & \quad \text{beam splitter}
\end{align*}
\]

Structure of PROP

by means of:

\[
\begin{align*}
D_1 \circ D_2 & \quad \text{sequential composition} \\
D_1 \oplus D_2 & \quad \text{parallel composition}
\end{align*}
\]
LOPP-Calculus: A Graphical Language for Linear Optical Circuits

LOPP-circuits are generated by:

\[ e^{i\varphi} \]

\[ \begin{pmatrix} \cos \theta & i \sin \theta \\
                i \sin \theta & \cos \theta \end{pmatrix} \]

together with:

\[
\begin{pmatrix} 0 & 1 \\
                1 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} [D_1] & 0 \\
                 0 & [D_2] \end{pmatrix}
\]

by means of:

sequential composition \( D_2 \circ D_1 \)

parallel composition \( D_1 \oplus D_2 \)

Structure of PROP
Universality of LOPP-Circuits

**Proposition**

For any unitary $U \in \mathbb{C}^{n \times n}$, there exists a LOPP-circuit $D$ such that $[D] = U$. 

Reck *et al.* (1994)  

Clements *et al.* (2016)
Complete Equational Theory for LOPP-Circuits

Proposition (Soundness)
\[ \forall D_1, D_2, \text{ if } \text{LOPP} \vdash D_1 = D_2 \text{ then } [D_1] = [D_2]. \]

Theorem (Completeness)
\[ \forall D_1, D_2, \text{ if } [D_1] = [D_2] \text{ then } \text{LOPP} \vdash D_1 = D_2. \]
Complete Equational Theory for LOPP-Circuits

\[ 0 = 2\pi = 0 \]

\[ 0 = \pi - \pi = \frac{\pi}{2} - \frac{\pi}{2} \]

\[ \gamma_1, \gamma_2, \gamma_4 = \delta_2, \delta_4, \delta_7 \]

\[ \gamma_3 = \delta_3, \delta_5, \delta_8, \delta_9 \]

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Theorem (Completeness)
\[ \forall D_1, D_2, \text{ if } [D_1] = [D_2] \text{ then } LOPP \vdash D_1 = D_2. \]
Proof of Completeness

**Proposition (Universality of LOPP)**
For any unitary $U \in \mathbb{C}^{n \times n}$, there exists a LOPP-circuit $D$ such that $[D] = U$.

**Proposition (Universality of QC)**
For any unitary $U \in \mathbb{C}^{2^n \times 2^n}$, there exists a quantum circuit $C$ such that $[C] = U$. 

n-Qubit Quantum Circuits

$C_1$

$C_2$

2$^n$-Mode Optical Circuits
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For any unitary \( U \in \mathbb{C}^{2^n \times 2^n} \), there exists a quantum circuit \( C \) such that \( [C] = U \).
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For any unitary $U \in \mathbb{C}^{2^n \times 2^n}$, there exists a quantum circuit $C$ such that $[C] = U$.  

n-Qubit Quantum Circuits

2$^n$-Mode Optical Circuits

Completeness of LOPP
Lemma 1

For any quantum circuit $C$, $\text{QC} \vdash D(E(C)) = C$.

Lemma 2

For every equation of the LOPP-calculus, of the form $D_1 = D_2$, one has $\text{QC} \vdash D(D_1) = D(D_2)$.
Proof of Completeness

Lemma 1
For any quantum circuit $C$, $\text{QC} \vdash D(E(C)) = C$.

Lemma 2
For every equation of the LOPP-calculus, of the form $D_1 = D_2$, one has $\text{QC} \vdash D(D_1) = D(D_2)$. 

2$^n$-Mode Optical Circuits

Completeness of LOPP
Key Aspects of the Proof

- Note that optical circuits and quantum circuits have different structures:

  Optical circuits: \[ \vdots D_1 \vdots \]
  \[ \vdots D_2 \vdots \]
  \[ D_1 \oplus D_2 \]

  Quantum circuits: \[ \vdots C_1 \vdots \]
  \[ \vdots C_2 \vdots \]
  \[ C_1 \otimes C_2 \]

- Decoding produces multi-controlled gates, e.g.:

  \[
  D \begin{pmatrix}
  \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots \\
  \end{pmatrix}
  =
  R_X(-2\theta)
  \]

- To define \( E \) and \( D \), we “sequentialise” the circuits:

  \[ \vdots C_1 \vdots = \vdots C_1 \vdots \]
  \[ \vdots C_2 \vdots = \vdots C_2 \vdots \]

- Some deformation rules need to be treated as proper equations.
Key Aspects of the Proof

- Note that optical circuits and quantum circuits have different structures:

  Optical circuits:
  \[
  \begin{array}{c}
  \vdots \\
  D_1 \\
  \vdots \\
  \vdots \\
  D_2 \\
  \vdots \\
  \vdots \\
  \vdots \\
  \end{array}
  \]

  \[D_1 \oplus D_2\]

  Quantum circuits:
  \[
  \begin{array}{c}
  \vdots \\
  C_1 \\
  \vdots \\
  \vdots \\
  C_2 \\
  \vdots \\
  \vdots \\
  \vdots \\
  \end{array}
  \]

  \[C_1 \otimes C_2\]

  \[\Rightarrow\]

  To define \(E\) and \(D\), we “sequentialise” the circuits:

  \[
  \begin{array}{c}
  \vdots \\
  C_1 \\
  \vdots \\
  \vdots \\
  C_2 \\
  \vdots \\
  \vdots \\
  \vdots \\
  \end{array} =
  \begin{array}{c}
  \vdots \\
  C_1 \\
  \vdots \\
  \vdots \\
  C_2 \\
  \vdots \\
  \vdots \\
  \vdots \\
  \end{array}
  \]

  \[\Rightarrow\]

  Some deformation rules need to be treated as proper equations.

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  \[
  D(f_{\theta}) = R_X(-2\theta)
  \]
Key Aspects of the Proof

- Note that optical circuits and quantum circuits have different structures:

  **Optical circuits:**
  - $D_1$
  - $D_2$

  $D_1 \oplus D_2$

  **Quantum circuits:**
  - $C_1$
  - $C_2$

  $C_1 \otimes C_2$

  ⇒ To define $E$ and $D$, we “sequentialise” the circuits:

  - $C_1$
  - $C_2$

  $C_1 \otimes C_2$

  $C_1 \oplus C_2$

  ⇒ Some deformation rules need to be treated as proper equations.

Decoding produces multi-controlled gates, e.g.:

$$
D \left( \begin{array}{c}
\theta
\end{array} \right) = R_X(-2\theta)
$$

A Complete Equational Theory for Quantum Circuits
Key Aspects of the Proof

- Note that optical circuits and quantum circuits have different structures:

  Optical circuits:
  \[
  D_1 \\
  D_2 \\
  D_1 \oplus D_2
  \]

  Quantum circuits:
  \[
  C_1 \\
  C_2 \\
  C_1 \otimes C_2
  \]

  \[D_1 \oplus D_2 \Rightarrow C_1 \otimes C_2\]

- To define \(E\) and \(D\), we “sequentialise” the circuits:

  \[
  \begin{align*}
  C_1 & = C_1 \\
  C_2 & = C_2
  \end{align*}
  \]

  \[\Rightarrow \text{Some deformation rules need to be treated as proper equations.}\]

- Decoding produces multi-controlled gates, e.g.:

  \[ D \begin{pmatrix}
      \shortmid \theta \\
      \shortmid \\
      \shortmid \\
    \end{pmatrix} = R_X(-2\theta) \]
Key Aspects of the Proof

Note that optical circuits and quantum circuits have different structures:

Optical circuits:

\[ D_1 \oplus D_2 \]

Quantum circuits:

\[ C_1 \otimes C_2 \]

⇒ To define \( E \) and \( D \), we “sequentialise” the circuits:

\[ C_1 \]

\[ C_2 \]

\[ \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

\[ \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

\[ = \]

\[ \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

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⇒ Some deformation rules need to be treated as proper equations.

Decoding produces multi-controlled gates, e.g.:

\[ D \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) = R_X(-2\theta) \]

\[ \left( \begin{array}{cccc}
1 & & & \\
& \cos \theta & i \sin \theta & \\
& i \sin \theta & \cos \theta & \\
& & & 1 \\
\end{array} \right) \]

two-level matrix
Future Work

- Simplify the equational theory, prove minimality
- Design procedures e.g. for circuit optimisation
- Prove upper and lower bounds on the size of the derivations, and of the intermediate circuits
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