EVIDENTIAL DECISION THEORY VIA PARTIAL MARKOV CATEGORIES

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PARTIALITY FOR OBSERVATIONS

Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.

\[ x \xrightarrow{\text{model}} A \]

Constraints cannot be total computations because \( \Rightarrow \neq = \).
OVERVIEW

Combine Markov and cartesian restriction categories to express partial stochastic processes.

Add the discrete structure to express equality checking.

Morphisms $x \xrightarrow{\delta} A$ are partial stochastic channels

$\delta(a | x) = \text{“probability of a given } x\text{”}$

$\delta(\bot | x) = \text{“probability of failure”}$

A partial Markov category is a copy-discard category with quasi-total conditionals.
EXAMPLES: PARTIAL STOCHASTIC PROCESSES

Partial stochastic processes form a partial Markov category. Maybe monad on a Markov category.

THEOREM

- Every Markov category with conditionals and coproducts
- Some ugly technical conditions

\( \Rightarrow \text{Kl}(\cdot+1) \) is a partial Markov category.

EXAMPLES

- \( \text{Kl}(D(\cdot+1)) \) \( \Rightarrow \) finitary subdistributions
- \( \text{Kl}(\text{Giry}(\cdot+1)) \) \( \Rightarrow \) subdistributions on standard Borel spaces
**BAYES INVERSION & NORMALISATION**

**BAYES INVERSION**

\[ g^+_\sigma(b|a) = \frac{g(a|b)\sigma(b)}{\sum_{b' \in B} g(a|b')\sigma(b')} \]

\( g^+_\sigma \) is a Bayes inversion of \( g \) w.r.t. \( \sigma \)

**NORMALISATION**

\[ \bar{g}(a|x) = \frac{g(x|a)}{1 - g(\perp|a)} \]

\( \bar{g} \) is a normalisation of \( g \)

→ Both are particular cases of quasi-total conditionals.
A discrete partial Markov category is a copy-discard category with quasi-total conditionals and comparators.
SYNTHETIC BAYES THEOREM

A deterministic observation $a: I \to A$ from a prior $\sigma: I \to X$ through a channel $c: X \to A$ determines an update proportional to the Bayes inversion $c^\sigma$ evaluated on $a$.

$$P(X=x|A=a) = \frac{P(A=a|X=x) \cdot P(X=x)}{\sum_{y \in X} P(A=a|X=y) \cdot P(X=y)}$$

classical formula for Bayes theorem
SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^r_\sigma$ evaluated on $a$.

PROOF
Processes with exact observations

For a Markov category $\mathcal{C}$ with conditionals, we construct a partial Markov category $\text{exOb}(\mathcal{C})$.

$$\text{exOb}(\mathcal{C}) = (\mathcal{C} + \{\alpha \rightarrow \alpha | \alpha \text{-} \alpha \text{ deterministic}\}) /$$

Conditionals and normalisations are computed in $\mathcal{C}$

normalisation of $\overline{\delta}$

conditional of $\overline{\delta}$
SUMMARY

Discrete partial Markov categories express stochastic processes with observations and updates.

\[
\sigma \cdot c \xrightarrow{a} x = \sigma \cdot c \xrightarrow{a} x
\]

**Synthetic Bayes theorem**

They are copy-discard categories with conditionals and comparators.
NEWCOMB’S PROBLEM

I predict that the agent will...

"One-box" \( \Rightarrow x = 10000 \)

"Two-box" \( \Rightarrow x = 0 \)

Should I "One-box" or "Two-box"?

Predictor

Opaque box with \( x \in \mathbb{R} \)

Transparent box with 1€

Agent

Very accurate: it is right 90\% of the times
**EVIDENTIAL DECISION THEORY**

Evidential decision theory answers:

"Which action would be **evidence** for the best-case scenario?"

My action is evidence for the prediction:

if I one-box I expect 10,000 €,
if I two-box I expect 1 €.

⇒ I will one-box

<table>
<thead>
<tr>
<th>Agent Predictor</th>
<th>One-box</th>
<th>Two-box</th>
</tr>
</thead>
<tbody>
<tr>
<td>One-box</td>
<td>10,000 €</td>
<td>10001 €</td>
</tr>
<tr>
<td>Two-box</td>
<td>0 €</td>
<td>1 €</td>
</tr>
</tbody>
</table>

*Most likely*
NEWCOMB’S PROBLEM CATEGORICALLY

Noise: the predictor is right 90% of the times

Utility function

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</tr>
</thead>
<tbody>
<tr>
<td>ONE-BOX</td>
<td></td>
<td>10 000 €</td>
<td>10 001 €</td>
</tr>
<tr>
<td>TWO-BOX</td>
<td></td>
<td>0 €</td>
<td>1 €</td>
</tr>
</tbody>
</table>
Evidential decision theory asks: “Which action would be evidence for the best-case scenario?” i.e. “Which action maximises the average of the state below?”