Formalizing $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda

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- The theorem is old but the setting is new: Brunerie's proof is carried out in *Homotopy Type Theory* (HoTT)
  - An extension of Martin-Löf type theory with *higher inductive types* (HITs) and *univalence*
- In addition to Bruneries proof, we present a new and vastly simplified proof.
- All formalisations are carried out in Cubical Agda
  - An extension of standard Agda supporting HITs and univalence.

Why?

- Brunerie's theorem is to this date one of the most advanced pieces of mathematics developed in HoTT
- Contains small 'gaps' which have made the theorem considered practically 'unformalisable'
- Let us start with a brief overview of Brunerie's original proof.

Chapters 1–3

- Brunerie constructs a map $[i_2, i_2] : \mathbb{S}^3 \to \mathbb{S}^2$.

$$\mathbb{S}^1 * \mathbb{S}^1$$

$$\mathbb{S}^3 \qquad\qquad \mathbb{S}^2$$

$$\mathbb{S}^2 \vee \mathbb{S}^2$$

[Animation]

Chapters 1–3

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- Define $\beta : \mathbb{Z}$ by $\beta = \psi[i_2, i_2]$
- **Main theorem:** We have $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$.

Chapters 1–3

What's needed?

- The long exact sequence of homotopy groups
- The James construction[1]
- The Hopf fibration
- The Blakers-Massey theorem[2]
- Whitehead products

Not easy, but doable!

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[1] We only formalised a special case

[2] Formalised by Kang Rongji

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- How to proceed?

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Brunerie's proof	**New proof**	**Normalise β**

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- Idea: ψ is unique up to a sign, so we may replace it with any other iso.
- We will divide it up into several steps, so that we can carefully trace $[i_2, i_2]$.

$$\pi_3(\mathbb{S}^2) \xrightarrow{F^*} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\mathsf{h}_*)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1)$$

$$\xrightarrow{F_*} \pi_3^*(\mathbb{S}^3) \xrightarrow{(F^{-1})^*} \pi_3(\mathbb{S}^3) \xrightarrow{\xi} \mathbb{Z}$$

Notation

$$\sigma : \mathbb{S}^1 \to \Omega(\mathbb{S}^2)$$

- Let $\smile : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ be the canonical map inducing the equivalence $\mathbb{S}^1 \wedge \mathbb{S}^1 \simeq \mathbb{S}^2$.

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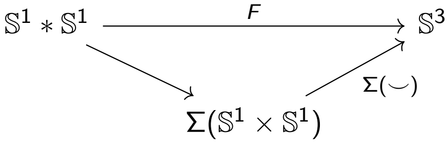

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- $\pi_3^*(A)$ can be given an explicit group structure s.t. $F^* : \pi_3(A) \cong \pi_3^*(A)$ (natural in A)

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- We then simply check that $\eta_1 \mapsto \eta_2 \mapsto \eta_3 \mapsto (-2)i_3$
  - Simple by since all isos (or their inverses) are decribed in terms of $\smile$ and h.

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Summary

- We have formalised $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ in three different ways:
  - Following Brunerie's original proof
  - Using the new simplified proof
  - By normalising a simplified Brunerie number
- All proofs rely on chapters 1–3 in Brunerie's thesis:
  - Future/ongoing work: can this part also be replaced by something simpler?