Cartesian Coherent Differential Categories

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IRIF

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Plan

1. Differential $\lambda$-calculus and (Cartesian) Differential Categories
2. Coherent differentiation
3. Sum and differentiation in a partial setting
4. Compatibility with the Cartesian product
5. Conclusion and perspectives
A function $f : E \to F$ is differentiable in $x$ if

$$f(x + u) \simeq f(x) + f'(x) \cdot u$$

With $f'(x) : E \to F$ a linear map.

Taylor Expansion

$$(\lambda x. P) Q \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^n P}{\partial x^n} \right) (Q, \ldots, Q)_{\text{\scriptsize $n$ times}}$$
Differential $\lambda$-calculus

A function $f : E \to F$ is differentiable in $x$ if

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With $f'(x) : E \to F$ a linear map.

Differential in terms

If $\Gamma, x : A \vdash P : B$ and $\Gamma \vdash Q : A$

$$\Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B$$

substitute one occurrence of $x$ by $Q$ in $P$. 
Differential \( \lambda \)-calculus

A function \( f : E \to F \) is differentiable in \( x \) if

\[
  f(x + u) \simeq f(x) + f'(x) \cdot u
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With \( f'(x) : E \to F \) a linear map.

Differential in terms

If \( \Gamma, x : A \vdash P : B \) and \( \Gamma \vdash Q : A \)

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  \Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B
\]

substitute one occurrence of \( x \) by \( Q \) in \( P \).

Taylor Expansion

\[
  (\lambda x. P)Q \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^n P}{\partial x^n} \cdot (Q, \ldots, Q) \right) \left[ 0/x \right]
\]
Recall: $L!(X, Y) := L(!X, Y)$ is a CCC.

1. Differential Category \( L \) (Linear Logic)
2. Cartesian Differential Category \( C \) (first order \( \lambda \)-calculus)

Kleisli ▶ Compatibility with the CCC structure: models of differential \( \lambda \)-calculus

Bucciarelli, Ehrhard, and Manzonetto 2010

▶ Models for Taylor expansion (qualitative setting) Manzonetto 2012

Example: relation model

Blute, Cockett, and Seely 2006.

Blute, Cockett, and Seely 2009.
Recall: $\mathcal{L}_!(X, Y) := \mathcal{L}(!X, Y)$ is a CCC

\begin{tikzcd}
\text{Differential Category}^1 \mathcal{L} \arrow[swap]{r}{\text{Kleisli}} & \text{Cartesian Differential Category}^2 \mathcal{C}
\end{tikzcd}

(Linear Logic) (first order $\lambda$-calculus)

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$^1$Blute, Cockett, and Seely 2006.

Recall: $\mathcal{L}(X, Y) := \mathcal{L}(!X, Y)$ is a CCC

Differential Category\(^1\) $\mathcal{L}$ (Linear Logic) \quad \xrightarrow{\text{Kleisli}} \quad$ Cartesian Differential Category\(^2\) $\mathcal{C}$ (first order $\lambda$-calculus)

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Cartesian Differential Category\(^2\) $\mathcal{C}$
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Cartesian Differential Category\(^2\) $\mathcal{C}$
(first order $\lambda$-calculus)

- Compatibility with the CCC structure: models of differential $\lambda$-calculus
  Bucciarelli, Ehrhard, and Manzonetto 2010
- Models for Taylor expansion (qualitative setting) Manzonetto 2012

Example: relation model

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\(^1\)Blute, Cockett, and Seely 2006.
\(^2\)Blute, Cockett, and Seely 2009.
(Left) additivity and non determinism

Leibniz: $f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v$
(Left) additivity and non determinism

Leibniz: \( f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v \)

A Differential Category \( \mathcal{L} \) must be additive

- \( \mathcal{L}(X, Y) \) is a commutative monoïd
- \((f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g \) (left additive)
- \( h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2 \) (additive)
(Left) additivity and non determinism

Leibniz: \( f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v \)

A Cartesian Differential Category \( \mathcal{C} \) must be left additive

- \( \mathcal{C}(X, Y) \) is a commutative monoïd
- \( (f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g \) (left additive)
- \( h \circ (f_1 + f_2) = h \circ f_1 + h \circ f_2 \) (additive)
(Left) additivity and non determinism

Leibniz: \( f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v \)

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Non-deterministic: \( \text{true}, \text{false} \in \mathcal{L}(1, 1 \oplus 1) \). What is \( \text{true} + \text{false} \) ?
(Left) additivity and non determinism

\[
\text{Leibniz: } f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v
\]

A Differential Category \( \mathcal{L} \) must be additive

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Non-deterministic: true, false \( \in \mathcal{L}(1, 1 \oplus 1) \). What is true + false ?

- If \((\lambda x. P)Q\) is well typed and reduces to a variable: only one member of \( \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^n P}{\partial x^n} \cdot (Q, \ldots, Q) \right) \) \( [0/x] \) is non zero.
(Left) additivity and non determinism

Leibniz: \( f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v \)

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- If \( (\lambda x . P)Q \) is well typed and reduces to a variable: only one member of \( \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^n P}{\partial x^n} \cdot (Q, \ldots, Q) \right) [0/x] \) is non zero.

- Interesting models \( \mathcal{L} \) of LL in which \( \mathcal{L}_1 \) is a category with differentiable morphisms, with a partial addition
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Our work in this paper

Differential Category\(^3\) (Linear Logic)

\[\xrightarrow{\text{Kleisli}}\]

Cartesian Differential Category\(^5\) (first order $\lambda$-calculus)

\[\xrightarrow{\text{Generalizes}}\]

Coherent Differential Category\(^4\) (Linear Logic)

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\(^3\)Blute, Cockett, and Seely 2006.  
\(^4\)Ehrhard 2023.  
\(^5\)Blute, Cockett, and Seely 2009.
Our work in this paper

Differential Category\(^3\) (Linear Logic)

Generalizes

Coherent Differential Category\(^4\) (Linear Logic)

Kleisli

Cartesian Differential Category\(^5\) (first order \(\lambda\)-calculus)

Generalizes

Cartesian Coherent Differential Category (first order \(\lambda\)-calculus)

\(^3\)Blute, Cockett, and Seely 2006.
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Our work in this paper

Differential Category\(^3\) (Linear Logic) \quad \text{Generalizes} \quad \text{Coherent Differential Category}\(^4\) (Linear Logic)

Kleisli

Cartesian Differential Category\(^5\) (first order λ-calculus) \quad \text{Generalizes} \quad \text{Cartesian Coherent Differential Category (first order λ-calculus)}

Models of a first order calculus with differentiation (subject reduction)

\(^3\)Blute, Cockett, and Seely 2006.
\(^4\)Ehrhard 2023.
\(^5\)Blute, Cockett, and Seely 2009.
Comparison with tangeant category

- Tangeant Category: distinguish point/vector, total sum on vectors
- Coherent Differential Category: no distinction point/vector, but restricted sum
Plan

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A categorical axiomatization of partial sum

A structure for partial sum

\( \tilde{D} : \text{Obj}(C) \to \text{Obj}(C) : \tilde{D}X = \{ \llangle x_0, x_1 \rrangle | x_0 + x_1 \text{ is defined} \} \)

- \( \pi_0, \pi_1 \in C(\tilde{D}X, X) \) jointly monic
A categorical axiomatization of partial sum

A structure for partial sum

\[ \tilde{D} : \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{C}) : \tilde{D}X = \{ \langle x_0, x_1 \rangle \mid x_0 + x_1 \text{ is defined} \} \]

- \( \pi_0, \pi_1 \in \mathcal{C}(\tilde{D}X, X) \) jointly monic
- Sum \( \sigma \in \mathcal{C}(\tilde{D}X, X) \) \( \sigma : \langle x_0, x_1 \rangle \mapsto x_0 + x_1 \)
A categorical axiomatization of partial sum

A structure for partial sum

\( \tilde{D} : \text{Obj}(C) \rightarrow \text{Obj}(C) : \tilde{D}X = \{ \langle x_0, x_1 \rangle | x_0 + x_1 \text{ is defined} \} \)

▶ \( \pi_0, \pi_1 \in C(\tilde{D}X, X) \) jointly monic

▶ Sum \( \sigma \in C(\tilde{D}X, X) \) \( \sigma : \langle x_0, x_1 \rangle \mapsto x_0 + x_1 \)

\( f_0, f_1 \in C(X, Y) \) \underline{summable}: \( \exists \langle f_0, f_1 \rangle \in C(X, \tilde{D}Y) \) s.t. \( \pi_i \circ \langle f_0, f_1 \rangle = f_i \).

\( x \mapsto \langle f_0(x), f_1(x) \rangle \)
A categorical axiomatization of partial sum

A structure for partial sum

\( \tilde{D} : \text{Obj}(C) \to \text{Obj}(C) : \tilde{D}X = \{x \mid x_0 + x_1 \text{ is defined} \} \)

- \( \pi_0, \pi_1 \in C(\tilde{D}X, X) \) jointly monic
- Sum \( \sigma \in C(\tilde{D}X, X) \) \( \sigma : x_0 + x_1 \mapsto x_0 + x_1 \)

\( f_0, f_1 \in C(X, Y) \) summable: \( \exists \langle f_0, f_1 \rangle \in C(X, \tilde{D}Y) \) s.t. \( \pi_i \circ \langle f_0, f_1 \rangle = f_i. \)

\( x \mapsto \langle f_0(x), f_1(x) \rangle \)

\[
\begin{array}{ccc}
X & \xrightarrow{\langle f_0, f_1 \rangle} & \tilde{D}Y \\
\downarrow & & \downarrow \sigma \\
f_0 + f_1 & & Y
\end{array}
\]
A categorical axiomatization of partial sum

A structure for partial sum

\[ \tilde{D} : \text{Obj}(C) \to \text{Obj}(C) : \tilde{D}X = \{ \langle x_0, x_1 \rangle | x_0 + x_1 \text{ is defined} \} \]

- \( \pi_0, \pi_1 \in \mathcal{C}(\tilde{D}X, X) \) jointly monic
- Sum \( \sigma \in \mathcal{C}(\tilde{D}X, X) \) \( \sigma : \langle x_0, x_1 \rangle \mapsto x_0 + x_1 \)

\( f_0, f_1 \in \mathcal{C}(X, Y) \) **summable**: \( \exists \langle f_0, f_1 \rangle \in \mathcal{C}(X, \tilde{D}Y) \) s.t. \( \pi_i \circ \langle f_0, f_1 \rangle = f_i \).

\[
\begin{array}{ccc}
X & \xrightarrow{\langle f_0, f_1 \rangle} & \tilde{D}Y \\
& & \sigma \\
f_0 + f_1 & \downarrow & Y
\end{array}
\]

\[ \tilde{D}X = X \& X (= X \times X) \iff \text{Cartesian Left Additive Category} \]
Summability structure

Compatibility with composition

If $g_0$ and $g_1$ are summable, then $g_0 \circ f$ and $g_1 \circ f$ are summable.

- $0 \circ f = 0$ and $(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f$ (left additive)
- $h \circ 0 = 0$ and $h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1$
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> $h \circ 0 = 0$ and $h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1$

Left summability structure

Axioms that endows $C(X, Y)$ with the structure of a partially additive monoid, see Arbib and Manes 1980
Summability structure

Compatibility with composition

If \( g_0 \) and \( g_1 \) are summable, then \( g_0 \circ f \) and \( g_1 \circ f \) are summable.

\[ 0 \circ f = 0 \text{ and } (g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f \text{ (left additive)} \]

\[ h \circ 0 = 0 \text{ and } h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1 \]

Left summability structure

Axioms that endows \( C(X, Y) \) with the structure of a partially additive monoid, see Arbib and Manes 1980

\( \tilde{D} \) is not a functor (yet)!
Differentiation

An operator for differentiation

Given $f \in \mathcal{C}(X, Y)$, there is $\tilde{D}f \in \mathcal{C}(\tilde{D}X, \tilde{D}Y)$ such that $\pi_0 \circ \tilde{D}f = f \circ \pi_0$

$$
\tilde{D}f : \tilde{D}X \rightarrow \tilde{D}Y
\langle x, u \rangle \mapsto \langle f(x), f'(x)u \rangle
$$
Differentiation

An operator for differentiation

Given $f \in C(X, Y)$, there is $\tilde{D}f \in C(\tilde{D}X, \tilde{D}Y)$ such that $\pi_0 \circ \tilde{D}f = f \circ \pi_0$

$$\tilde{D}f : \tilde{D}X \rightarrow \tilde{D}Y$$

$$\langle x, u \rangle \mapsto \langle f(x), f'(x).u \rangle$$

Define $f' = \pi_1 \circ \tilde{D}f \in C(\tilde{D}X, Y)$
Differentiation

An operator for differentiation

Given \( f \in C(X, Y) \), there is \( \tilde{D}f \in C(\tilde{D}X, \tilde{D}Y) \) such that \( \pi_0 \circ \tilde{D}f = f \circ \pi_0 \)

\[
\tilde{D}f : \quad \tilde{D}X \rightarrow \tilde{D}Y \\
\langle x, u \rangle \mapsto \langle f(x), f'(x).u \rangle
\]

Define \( f' = \pi_1 \circ \tilde{D}f \in C(\tilde{D}X, Y) \)

Axioms of differentiation: very structural properties

► \( \tilde{D} \) is a functor (Chain rule)
Differentiation

An operator for differentiation

Given \( f \in C(X, Y) \), there is \( \tilde{D}f \in C(\tilde{D}X, \tilde{D}Y) \) such that \( \pi_0 \circ \tilde{D}f = f \circ \pi_0 \)

\[
\tilde{D}f : \quad \tilde{D}X \rightarrow \tilde{D}Y \\
\langle x, u \rangle \mapsto \langle f(x), f'(x) \cdot u \rangle
\]

Define \( f' = \pi_1 \circ \tilde{D}f \in C(\tilde{D}X, Y) \)

Axioms of differentiation: very structural properties

- \( \tilde{D} \) is a functor (Chain rule)
- \( \pi_0, \pi_1 \) are linear (\( h \) linear if \( h'(x) \cdot u = h(u) \))
- \( \sigma \) is linear (\( (f + g)' = f' + g' \))
Differentiation

An operator for differentiation

Given \( f \in \mathcal{C}(X, Y) \), there is \( \tilde{D}f \in \mathcal{C}(\tilde{D}X, \tilde{D}Y) \) such that \( \pi_0 \circ \tilde{D}f = f \circ \pi_0 \)

\[
\tilde{D}f : \tilde{D}X \rightarrow \tilde{D}Y \\
\langle x, u \rangle \mapsto \langle f(x), f'(x).u \rangle
\]

Define \( f' = \pi_1 \circ \tilde{D}f \in \mathcal{C}(\tilde{D}X, Y) \)

Axioms of differentiation: very structural properties

- \( \tilde{D} \) is a functor (Chain rule)
- \( \pi_0, \pi_1 \) are linear (\( h \) linear if \( h'(x) \cdot u = h(u) \))
- \( \sigma \) is linear (\( (f + g)' = f' + g' \))
- Leibniz + Schwarz + the differential is linear = naturality!
Define $\nu_0$, $\theta$, $c$ and $l$

\[ \nu_0 \circ x = \langle x, 0 \rangle \]

\[ \theta \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle x, u + v \rangle \]

\[ c \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle \langle x, v \rangle, \langle u, w \rangle \rangle \]

\[ l \circ \langle x, u \rangle = \langle \langle x, 0 \rangle, \langle 0, u \rangle \rangle \]

- $\tilde{D}$ is a monad with unit $\nu_0$ and sum $\theta$ (The differential is additive $=$ Leibniz)
- $c$ is natural (Schwarz)
- $l$ is natural (The differential is linear)
Define \( \nu_0, \theta, c \) and \( l \)

\[
\nu_0 \circ x = \langle x, 0 \rangle \\
\theta \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle x, u + v \rangle \\
c \circ \langle \langle x, u \rangle, \langle v, w \rangle \rangle = \langle \langle x, v \rangle, \langle u, w \rangle \rangle \\
l \circ \langle x, u \rangle = \langle \langle x, 0 \rangle, \langle 0, u \rangle \rangle
\]

\( \tilde{D} \) is a monad with unit \( \nu_0 \) and sum \( \theta \) (The differential is additive = Leibniz)

\( c \) is natural (Schwarz)

\( l \) is natural (The differential is linear)

**Cartesian Differential Categories**

Naturality equations \( \iff \) equations on the differential \( f' \).

They are exactly the equations of Cartesian Differential Categories.

\[
\text{Cartesian Differential Category } \iff \tilde{D}X = X \& X
\]
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Compatibility with Cartesian Product

Compatibility with the Cartesian product

- Product and sum: $\langle x, y \rangle + \langle u, v \rangle = \langle x + y, u + v \rangle$

- Product and differential: the projections of the cartesian product are linear, $D\langle f, g \rangle = \langle Df, Dg \rangle$
Compatibility with Cartesian Product

Compatibility with the Cartesian product

- Product and sum: \( \langle x, y \rangle + \langle u, v \rangle = \langle x + y, u + v \rangle \)
- Product and differential: the projections of the cartesian product are linear, \( \text{D}\langle f, g \rangle = \langle \text{D}f, \text{D}g \rangle \)

In analysis (and Cartesian Differential Categories)

\[
\partial_0 f(x, y) \cdot u = f'(x, y) \cdot (u, 0)
\]
Compatibility with Cartesian Product

Compatibility with the Cartesian product

- Product and sum: \( \langle x, y \rangle + \langle u, v \rangle = \langle x + y, u + v \rangle \)
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In analysis (and Cartesian Differential Categories)

\[
\partial_0 f(x, y) \cdot u = f'(x, y) \cdot (u, 0)
\]

In our setting: strength \( \Phi^0 \in C(\tilde{D} X_0 \& X_1, \tilde{D}(X_0 \& X_1)) \)

\[
\Phi^0 : \tilde{D} X_0 \& X_1 \rightarrow \tilde{D} X_0 \& \tilde{D} X_1 \cong \tilde{D}(X_0 \& X_1)
\]

\[
\langle\langle x, u \rangle, y \rangle \mapsto \langle\langle x, u \rangle, \langle y, 0 \rangle \rangle \mapsto \langle\langle x, y \rangle, \langle u, 0 \rangle \rangle
\]
Compatibility with the Cartesian product

- Product and sum: \( \langle x, y \rangle + \langle u, v \rangle = \langle x + y, u + v \rangle \)
- Product and differential: the projections of the cartesian product are linear, \( D\langle f, g \rangle = \langle Df, Dg \rangle \)

In analysis (and Cartesian Differential Categories)

\[
\partial_0 f(x, y) \cdot u = f'(x, y) \cdot (u, 0)
\]

In our setting: strength \( \Phi^0 \in C(\tilde{D}X_0 & X_1, \tilde{D}(X_0 & X_1)) \)

\[
\Phi^0 : \quad \tilde{D}X_0 & X_1 \to \tilde{D}X_0 & \tilde{D}X_1 \simeq \tilde{D}(X_0 & X_1) \\
\langle\langle x, u \rangle, y \rangle \mapsto \langle\langle x, u \rangle, \langle y, 0 \rangle \rangle \mapsto \langle\langle x, y \rangle, \langle u, 0 \rangle \rangle
\]

Partial derivative of \( f \in C(X_0 & X_1, Y) \): \( \tilde{D}_0 f \in C(\tilde{D}X_0 & X_1, \tilde{D}Y) \)

\[
\tilde{D}X_0 & X_1 \xrightarrow{\Phi^0} \tilde{D}(X_0 & X_1) \xrightarrow{\tilde{D}f} \tilde{D}Y
\]
Leibniz and Schwarz

**Leibniz**

In analysis:

\[
f'(x, y) \cdot (u, v) = \partial_0 f(x, y) \cdot u + \partial_1 f(x, y) \cdot v
\]

In Cartesian Coherent Differential Categories

\[
\tilde{D} f \circ c^{-1} = \theta \circ \tilde{D}_0 \tilde{D}_1 f = \theta \circ \tilde{D}_1 \tilde{D}_0 f
\]
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Takeaway

- Axiomatization of differentiation with partial sums
- Axioms of differentiation: functoriality and naturality
- Nice theory of partial derivatives
TODO list

- Introduce closure to interpret a deterministic differential $\lambda$-calculus
- Deal with fixpoints to interpret the Coherent Differential PCF of Ehrhard
- Revisit syntactical Taylor expansion in a coherent setting
- It should provide generic denotational proofs of important results on syntactical Taylor expansion
- Is this construction insightful for traditional analysis?