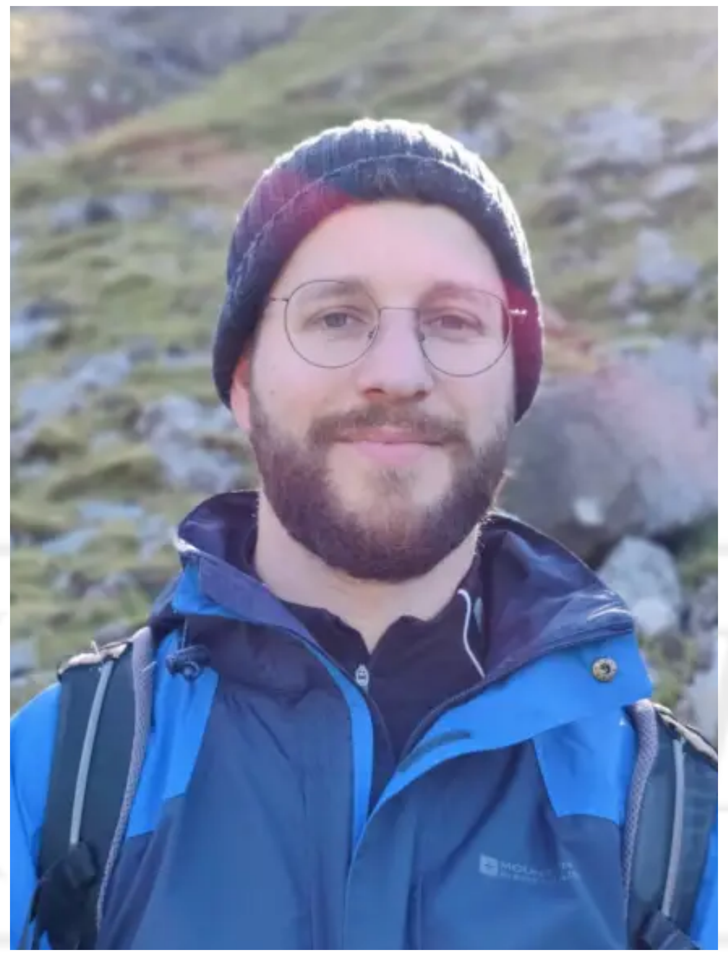

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Cut-restriction

From cuts to analytic cuts

LICS 2023

Our paper in a nutshell

- The analytic cut property (ACP) is a useful generalisation of cut-admissibility.
- We show how to ***constructively*** obtain the ACP for a class of logics, using an extension of Gentzen's cut-elimination algorithm.

Some Background

Cut-Elimination (G. Gentzen 1934)

- A technique for obtaining normal forms of proofs.
- Proceeds by eliminating instances of the ***cut rule***

$$\frac{\Gamma \Rightarrow C \quad C, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \; (\mathit{cut})$$

  from ***sequent calculus*** proofs.

- Cutfree proofs have the ***subformula property*** and are **very useful** for ***proof search*** and obtaining ***metatheorems*** (e.g. decidability, interpolation).

- Sequent calculi with cut-elimination:

K ✓ **KT** ✓ **KD** ✓ **S4** ✓ **LL** ✓ **ILL** ✓ **FL** ✓ **FLew** ✓ **…**

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- But: **S5** ✗ **B** ✗ **BiInt** ✗ other modal logics ✗
- How to deal with this?

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Generalise the sequent calculus

1^{st} Option:

Find a more complicated calculus in which cut-elimination works

- Hypersequent calculus
- Labelled sequent calculus
- Nested sequent calculus
- .....

- Sequent calculi with cut-elimination:

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Generalise the sequent calculus

1st Option:

Find a more complicated calculus in which cut-elimination works

- Hypersequent calculus
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- .....

Generalise cut-elimination

2nd Option:

Find a weaker property than cut-elimination that holds in the sequent calculus

e.g. analytic cut property

The analytic cut property (ACP)

- A cut $\dfrac{\Gamma \Rightarrow C \quad C, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}$ is **analytic** if $C \in subf(\Gamma, \Delta, \Pi)$.
  (cf. Smullyan 1968)
- A *proof* is analytic if all cuts in it are analytic ($\rightsquigarrow$ subformula property!).
- A sequent *calculus* has the **ACP** if the cut rule can be restricted to analytic cuts.

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The ACP is “almost as good as” cut-elimination.
But how do we prove it?

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But how do we prove it?

semantic (non-constructive): e.g.

T. Kowalski & H. Ono (2017):
Analytic cut and interpolation for bi-intuitionistic logic

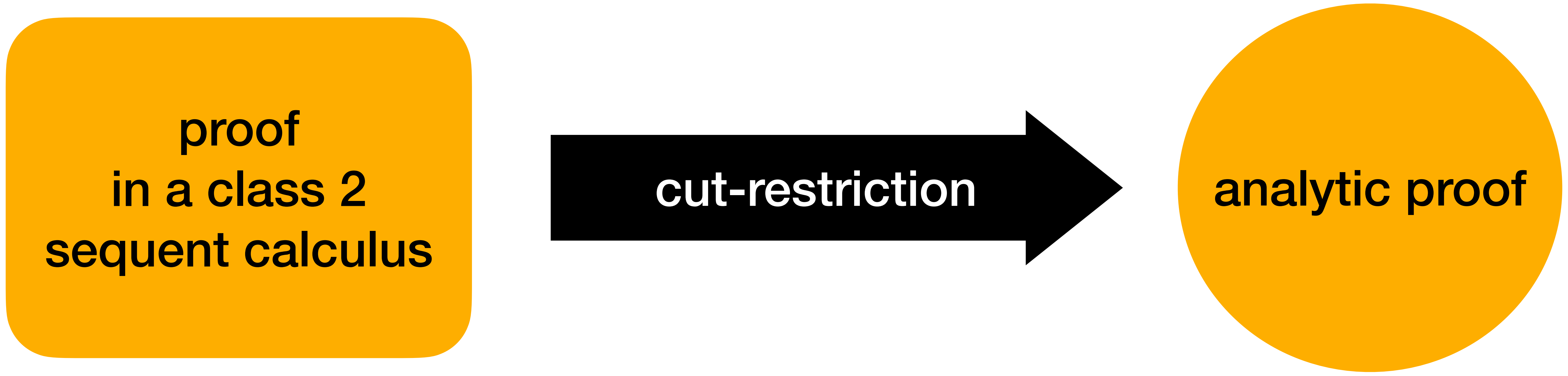
syntactic (constructive):

M. Takano (1991):
Subformula property as a substitute for cut-elimination in modal propositional logics

Cut-restriction

An algorithm for obtaining the ACP

- A set of simple-to-check conditions on sequent calculi: ***class 2***
- A uniform, constructive proof transformation: ***cut-restriction***



Main Theorem

Every class 2 calculus has the ACP.

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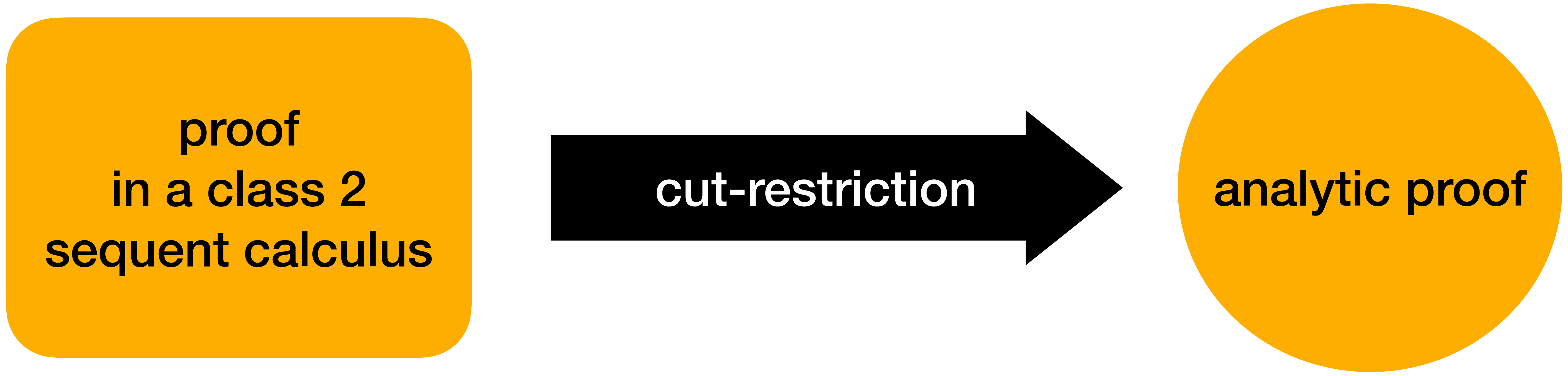

Main Theorem

Every class 2 calculus has the ACP.

This covers uniformly: **S5** (reproving Takano '91) **BiInt** (first syntactic proof of the ACP)

L4, **BiInt**(**S5**), multi-modal **S5, **…?

A glimpse into cut-restriction

All the usual local Gentzen reduction steps + one **global reduction**:

$$\dfrac{\begin{array}{c}\vdots\\ \dfrac{\Box A \Rightarrow C}{\Box A \Rightarrow \Box C}\\ \vdots\\ \Gamma \Rightarrow \Box C\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ C \Rightarrow D\end{array}}{\Box C \Rightarrow D}}{\Gamma \Rightarrow D}\,(cut) \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\dfrac{}{\Box A \Rightarrow \Box A}\\ \vdots\,\Box A/\Box C\\ \Gamma \Rightarrow \Box A\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \Box A \Rightarrow C\end{array} \quad \begin{array}{c}\vdots\\ C \Rightarrow D\end{array}}{\Box A \Rightarrow D}\,(cut)}{\Gamma \Rightarrow D}\,(cut)$$

- The global reduction involves a (global) **substitution**
- The substitution can make cuts non-analytic (repeat the procedure on those cuts!)

Two open questions

1. What is the ***computational interpretation*** of analytic cuts?

2. Generalising the sequent calculus VS generalising cut elimination
   — How do these approaches relate? (e.g. ***proof complexity***)

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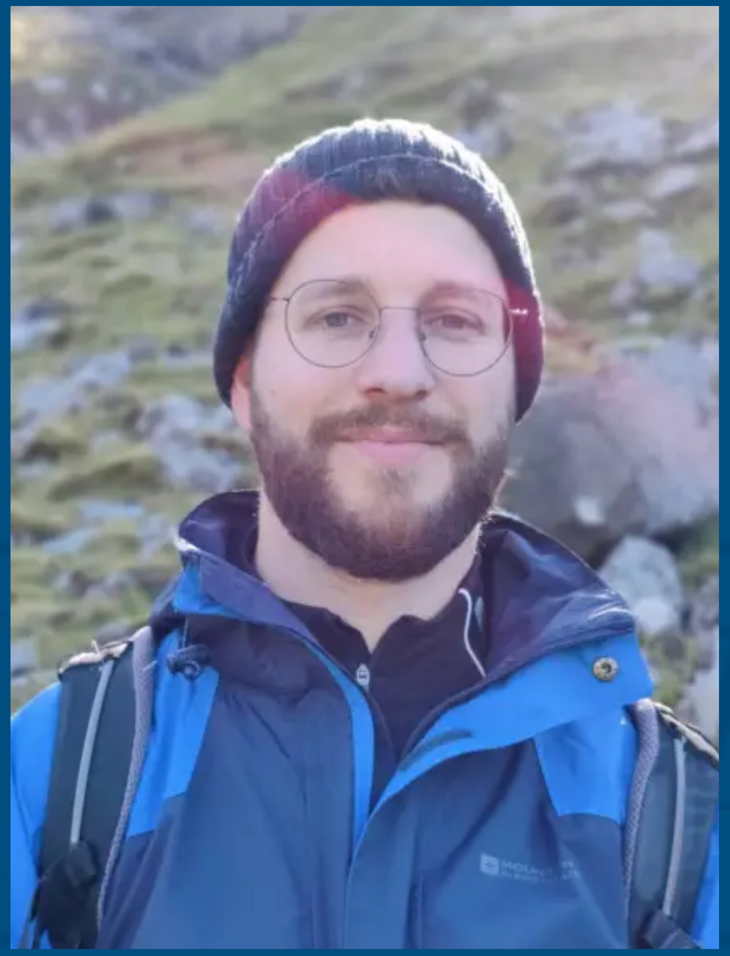

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- The analytic cut property is a useful generalisation of cut-elimination.
- We developed *cut-restriction*, an algorithm that rewrites arbitrary cuts into analytic cuts.
- The algorithm is logic-independent and works once suitable sufficient conditions are satisfied.