Cut-restriction

From cuts to analytic cuts

(LICS 2023)
Our paper in a nutshell

• The analytic cut property (ACP) is a useful generalisation of cut-admissibility.

• We show how to constructively obtain the ACP for a class of logics, using an extension of Gentzen’s cut-elimination algorithm.
Some Background

\[
\frac{(A \rightarrow B) \land (B \rightarrow C)}{A \rightarrow C}^2
\]

\[
\frac{B}{(A \rightarrow B) \land (B \rightarrow C)}
\]

\[
\frac{B \rightarrow C}{(A \rightarrow B) \land (B \rightarrow C) \rightarrow (A \rightarrow C)}^2
\]
Cut-Elimination (G. Gentzen 1934)

- A technique for obtaining normal forms of proofs.
- Proceeds by eliminating instances of the cut rule

\[
\frac{\Gamma \Rightarrow C \quad C, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \quad \text{(cut)}
\]

from sequent calculus proofs.

- Cutfree proofs have the subformula property and are very useful for proof search and obtaining metatheorems (e.g. decidability, interpolation).
• Sequent calculi with cut-elimination:

K ✓ KT ✓ KD ✓ S4 ✓ LL ✓ ILL ✓ FL ✓ FLew ✓ ...
• Sequent calculi with cut-elimination:

\[ K \checkmark \quad KT \checkmark \quad KD \checkmark \quad S4 \checkmark \quad LL \checkmark \quad ILL \checkmark \quad FL \checkmark \quad FLew \checkmark \quad \ldots \]

• But: \( S5 \times \quad B \times \quad Bilnt \times \quad \text{other modal logics} \times \)

• How to deal with this?
• Sequent calculi with cut-elimination:

  K ✓  KT ✓  KD ✓  S4 ✓  LL ✓  ILL ✓  FL ✓  FLew ✓  ...

• But: S5 ✗  B ✗  Bilnt ✗  other modal logics ✗

• How to deal with this?

  1st Option:

  Find a more complicated calculus in which cut-elimination works
  • Hypersequent calculus
  • Labelled sequent calculus
  • Nested sequent calculus
  • .....
• Sequent calculi with cut-elimination:

  K ✓ KT ✓ KD ✓ S4 ✓ LL ✓ ILL ✓ FL ✓ FLew ✓ ...

• But: S5 ✗ B ✗ Bilnt ✗ other modal logics ✗

• How to deal with this?

  1st Option:
  Find a more complicated calculus in which cut-elimination works
  • Hypersequent calculus
  • Labelled sequent calculus
  • Nested sequent calculus
  • …..

  2nd Option:
  Find a weaker property than cut-elimination that holds in the sequent calculus
  e.g. analytic cut property

Generalise the sequent calculus

Generalise cut-elimination
The analytic cut property (ACP)

• A cut \( \frac{\Gamma \Rightarrow C \quad C, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \) is \textbf{analytic} if \( C \in \text{subf}(\Gamma, \Delta, \Pi) \).

  (cf. Smullyan 1968)

• A \textit{proof} is analytic if all cuts in it are analytic (\( \rightsquigarrow \text{subformula property!} \)).

• A sequent \textit{calculus} has the \textbf{ACP} if the cut rule can be restricted to analytic cuts.
The analytic cut property (ACP)

- A cut
  \[ \Gamma \Rightarrow C \quad C, \Delta \Rightarrow \Pi \]
  is analytic if
  \[ C \in \text{subf}(\Gamma, \Delta, \Pi). \]
  (cf. Smullyan 1968)

- A proof is analytic if all cuts in it are analytic (subformula property!).

- A sequent calculus has the ACP if the cut rule can be restricted to analytic cuts.

The ACP is “almost as good as” cut-elimination.
But how do we prove it?
The analytic cut property (ACP)

• A cut $\Gamma \Rightarrow C, C, \Delta \Rightarrow \Pi$ is **analytic** if $C \in \text{subf}(\Gamma, \Delta, \Pi)$. (cf. Smullyan 1968)

• A *proof* is analytic if all cuts in it are analytic (≈ subformula property!).

• A sequent *calculus* has the **ACP** if the cut rule can be restricted to analytic cuts.

The ACP is “almost as good as” cut-elimination. But how do we prove it?

**semantic (non-constructive):** e.g.
T. Kowalski & H. Ono (2017):
Analytic cut and interpolation for bi-intuitionistic logic

**syntactic (constructive):**
M. Takano (1991):
Subformula property as a substitute for cut-elimination in modal propositional logics
Cut-restriction
An algorithm for obtaining the ACP

\[
\frac{(A \rightarrow B) \land (B \rightarrow C)}{B} \quad 2
\]

\[
\frac{(A \rightarrow B) \land (B \rightarrow C)}{B \rightarrow C}
\]

\[
\frac{C}{A \rightarrow C} \quad 1
\]

\[
\frac{(A \rightarrow B) \land (B \rightarrow C)}{(A \rightarrow C)} \quad 2
\]
• A set of simple-to-check conditions on sequent calculi: \textit{class 2}
• A uniform, constructive proof transformation: \textit{cut-restriction}

\textbf{Main Theorem}

Every class 2 calculus has the ACP.
• A set of simple-to-check conditions on sequent calculi: \textit{class 2}

• A uniform, constructive proof transformation: \textit{cut-restriction}

\begin{align*}
\text{proof in a class 2 sequent calculus} & \xrightarrow{\text{cut-restriction}} \text{analytic proof} \\
\text{Main Theorem} & \\
\text{Every class 2 calculus has the ACP.}
\end{align*}

This covers uniformly: \textbf{S5} (reproving Takano ’91) \hspace{1cm} \textbf{BiInt} (first syntactic proof of the ACP)

\textbf{L4, BiInt(S5), multi-modal S5, …?}
A glimpse into cut-restriction

All the usual local Gentzen reduction steps + one **global reduction**:

- The global reduction involves a (global) **substitution**
- The substitution can make cuts non-analytic (repeat the procedure on those cuts!)
Two open questions

\[(A \rightarrow B) \land (B \rightarrow C)\]
1. What is the computational interpretation of analytic cuts?

2. Generalising the sequent calculus VS generalising cut elimination
   — How do these approaches relate? (e.g. proof complexity)
The **analytic cut property** is a useful generalisation of cut-elimination.

We developed **cut-restriction**, an algorithm that rewrites arbitrary cuts into analytic cuts.

The algorithm is logic-independent and works once suitable sufficient conditions are satisfied.