Quantifying over Trees in Monadic Second-Order Logics

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(joint work with Massimo Benerecetti, Laura Bozzelli, & Adriano Peron)

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Monadic Second-Order Logic

- MSOL = FOL + Quantification over Sets of Elements

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    - Undecidable over arbitrary structures
    - Decidable on trees [Rabin'69]
    - Decidable on graphs with bounded-tree width [Courcelle'89]
    - Decidable on tree-like structures

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- Sets of Computations viewed as Trees

  ↓

  Relation with Temporal Logics

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Relation with Temporal Logics

$$\begin{tikzcd}
\mu\text{Calculus} \arrow[r, hook] & \text{MSOL} \\
\text{CTL}^* \arrow[u] \arrow[r, hook] & \text{MPL} \arrow[u]
\end{tikzcd}$$

MSOL: Quantification on arbitrary sets of nodes

MPL: Quantification on paths (connected linear orders)

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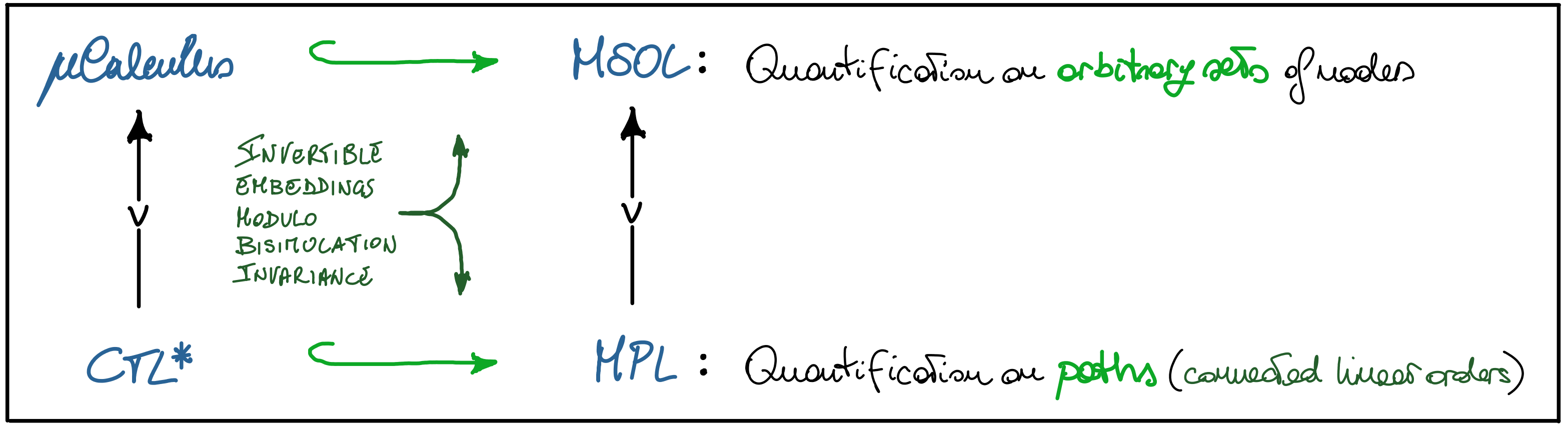

μCalculus ⟶ MSOL: Quantification on arbitrary sets of nodes

CTL* ⟶ MPL : Quantification on paths (connected linear orders)

A Closer Look at the Embeddings

μCalculus $\hookrightarrow$ MSOL

$(E q R_2 p \equiv)$ $\nu X.(p \wedge (q \vee \Diamond\Diamond X))$

$(\nu X.(p \wedge (q \vee \Diamond X)) \equiv)$ $E q R p$

CTL* $\hookrightarrow$ MPL

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μCalculus $\hookrightarrow$ MSOL

$(E\,qR_2p \equiv)$	$\nu X.(p \wedge (q \vee \Diamond\Diamond X))$	p	*	p	*	p	*	p	*	p,q	*	
		0	1	2	3	4	5	6	7	8	9	…
$(\nu X.(p \wedge (q \vee \Diamond X)) \equiv)$	$E\,qRp$	p	p	p	p	p,q	*	p	p	p	p,q	

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$\forall x \in X.\ p(x) \wedge \left(\overset{q(x)\ \vee}{\exists y \exists z. R(x,y) \wedge R(y,z) \wedge z \in X} \right)$

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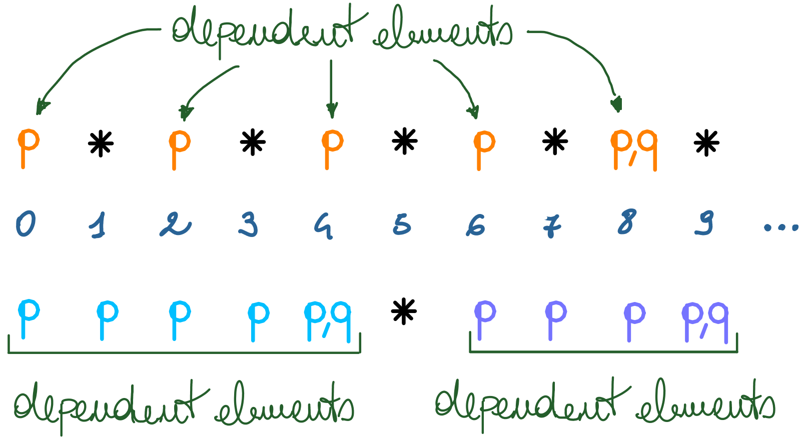

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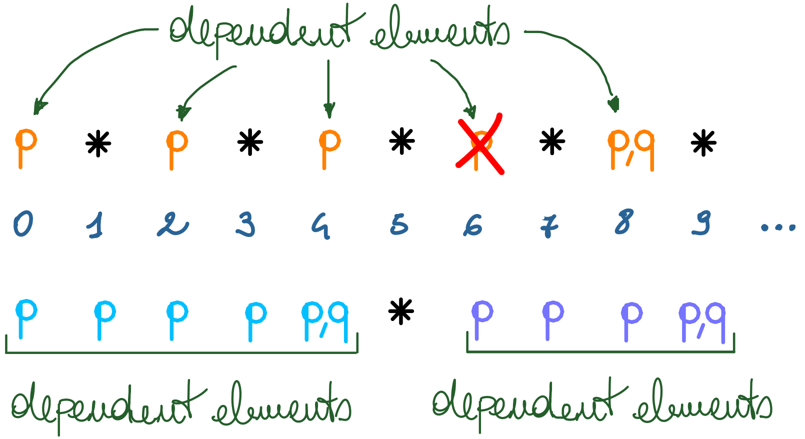

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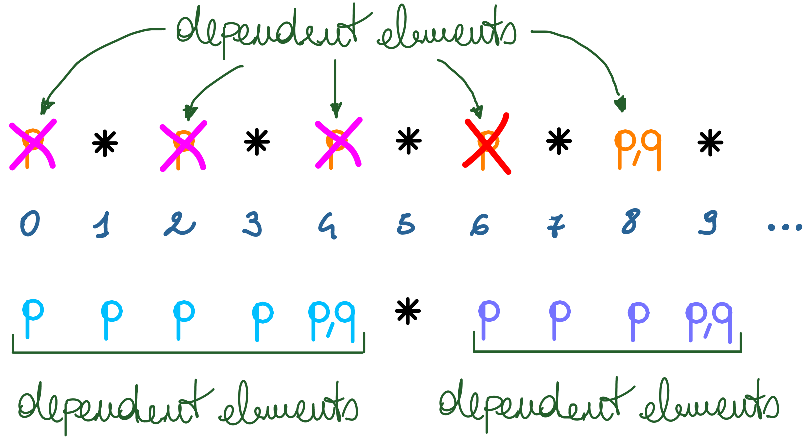

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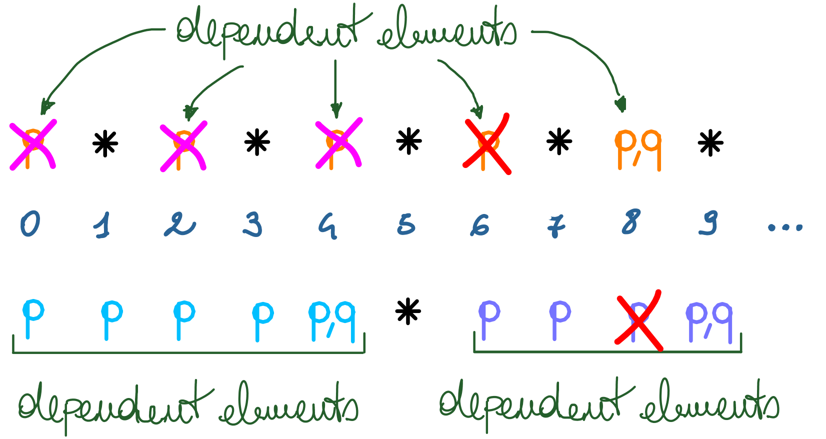

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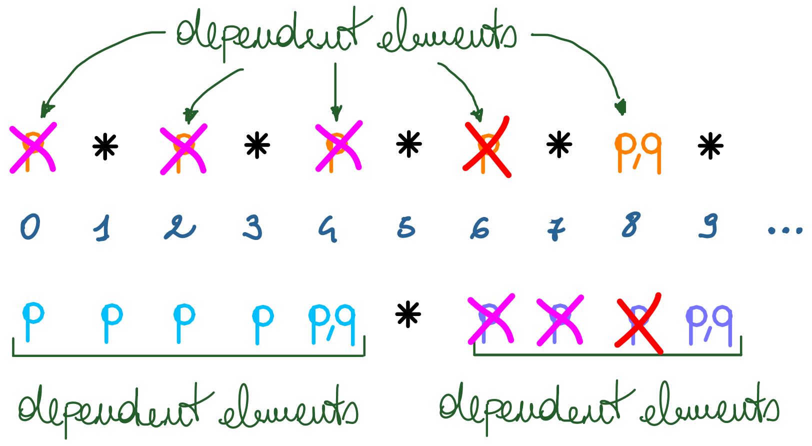

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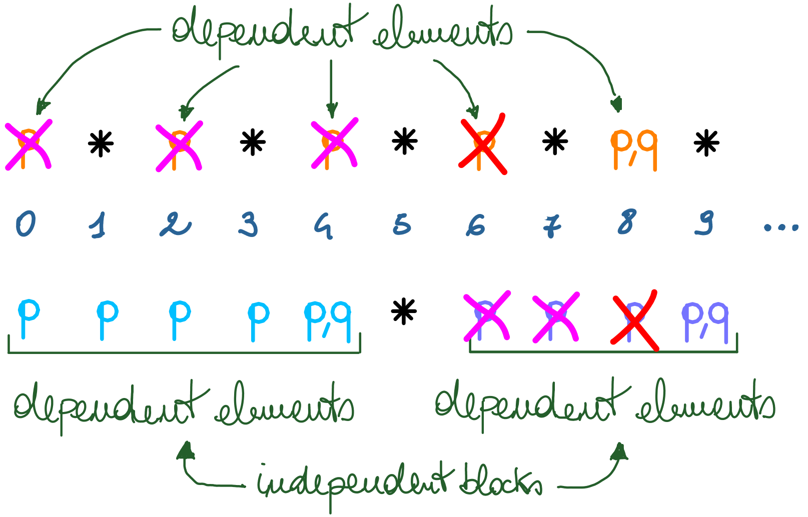

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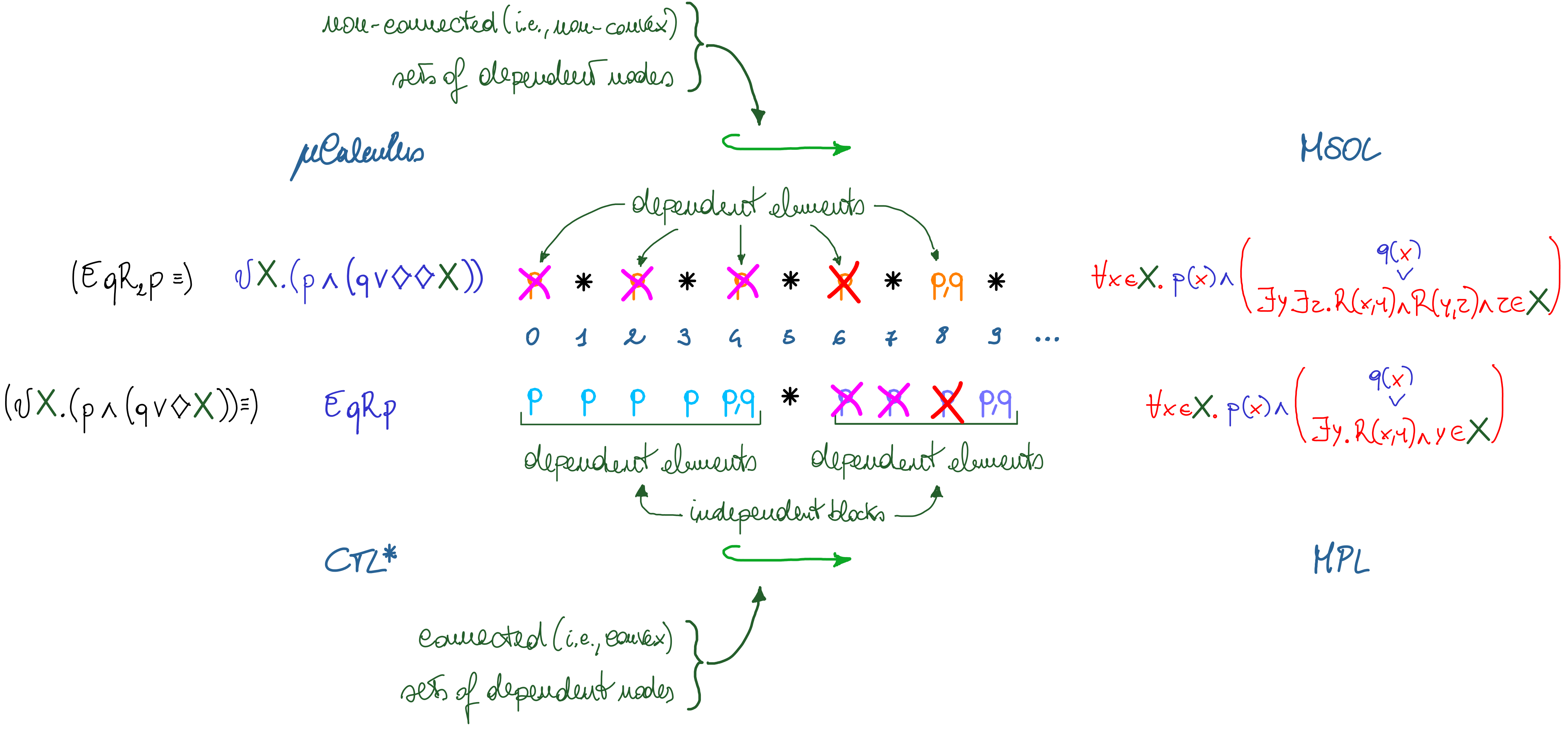

non-connected (i.e., non-convex) sets of dependent nodes

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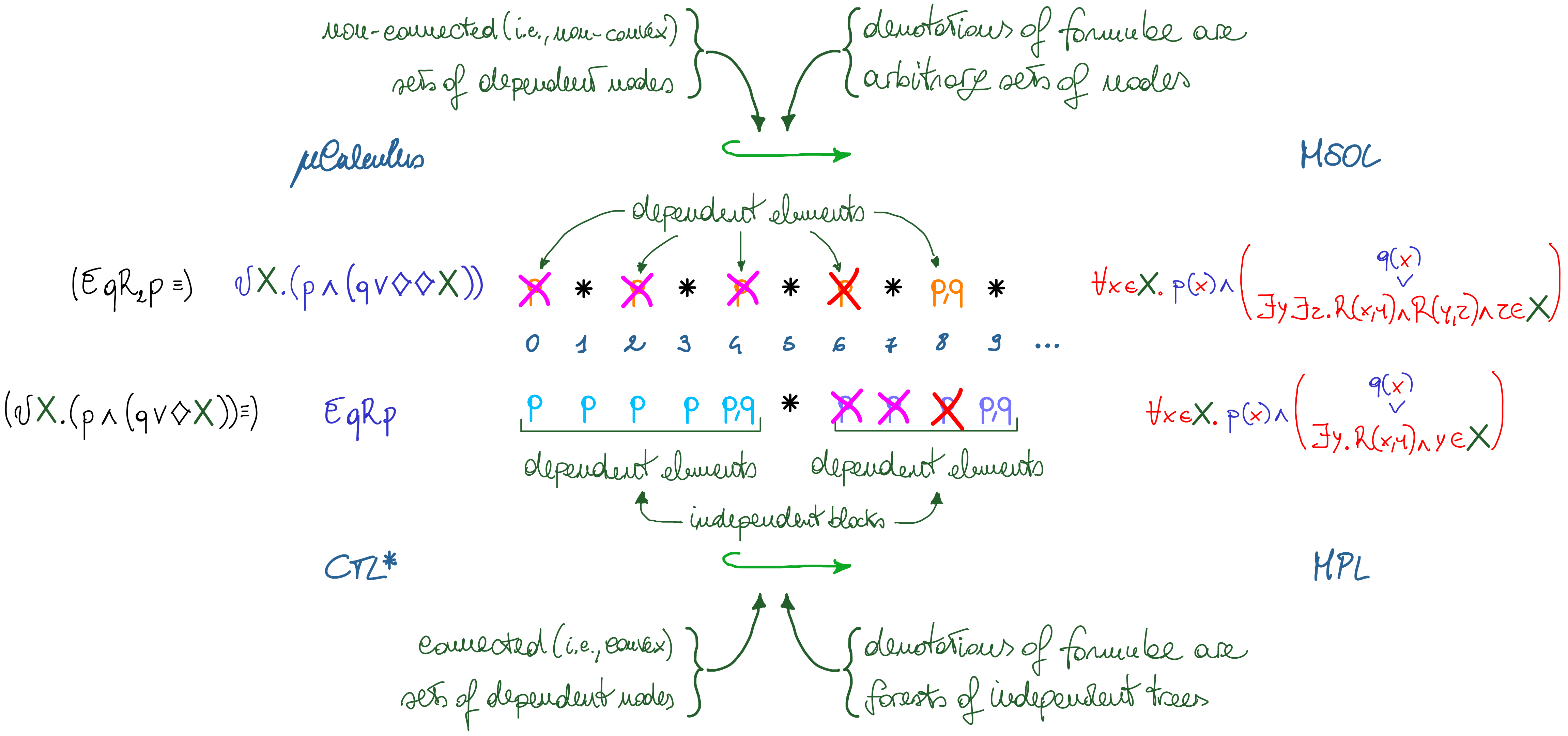

non-connected (i.e., non-convex) sets of dependent nodes

denotations of formulae are arbitrary sets of nodes

μCalculus $\hookrightarrow$ MSOL

dependent elements

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$\forall x \in X.\ p(x) \wedge \left(\exists y \exists z. R(x,y) \wedge R(y,z) \wedge z \in X \right)$ with $q(x)\ \vee$ above

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dependent elements

dependent elements

independent blocks

CTL* $\hookrightarrow$ MPL

connected (i.e., convex) sets of dependent nodes

denotations of formulae are forests of independent trees

A Missing Piece: Quantification over Trees

- Connectedness of denotation holds for other logics as well: $\underbrace{\text{ATL}^*,\ \text{SL},\ \text{STL}^*}_{\text{quantifications over strategies or substructures boil down to quantifications over subtrees}}$, etc.

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* A natural fragment: Monadic Tree Logic (MTL) = MSOL with quantifications over trees

- An easy observation: MTL < MSOL!
- A harder question: Does quantification over trees reduce to quantification over paths, namely MPL < MTL?

Our Contribution

* Expressive comparisons among fragments of MSOL

  → Introduction of new fragments of MSOL called "co-Weak"

  → Analysis of interesting separation properties

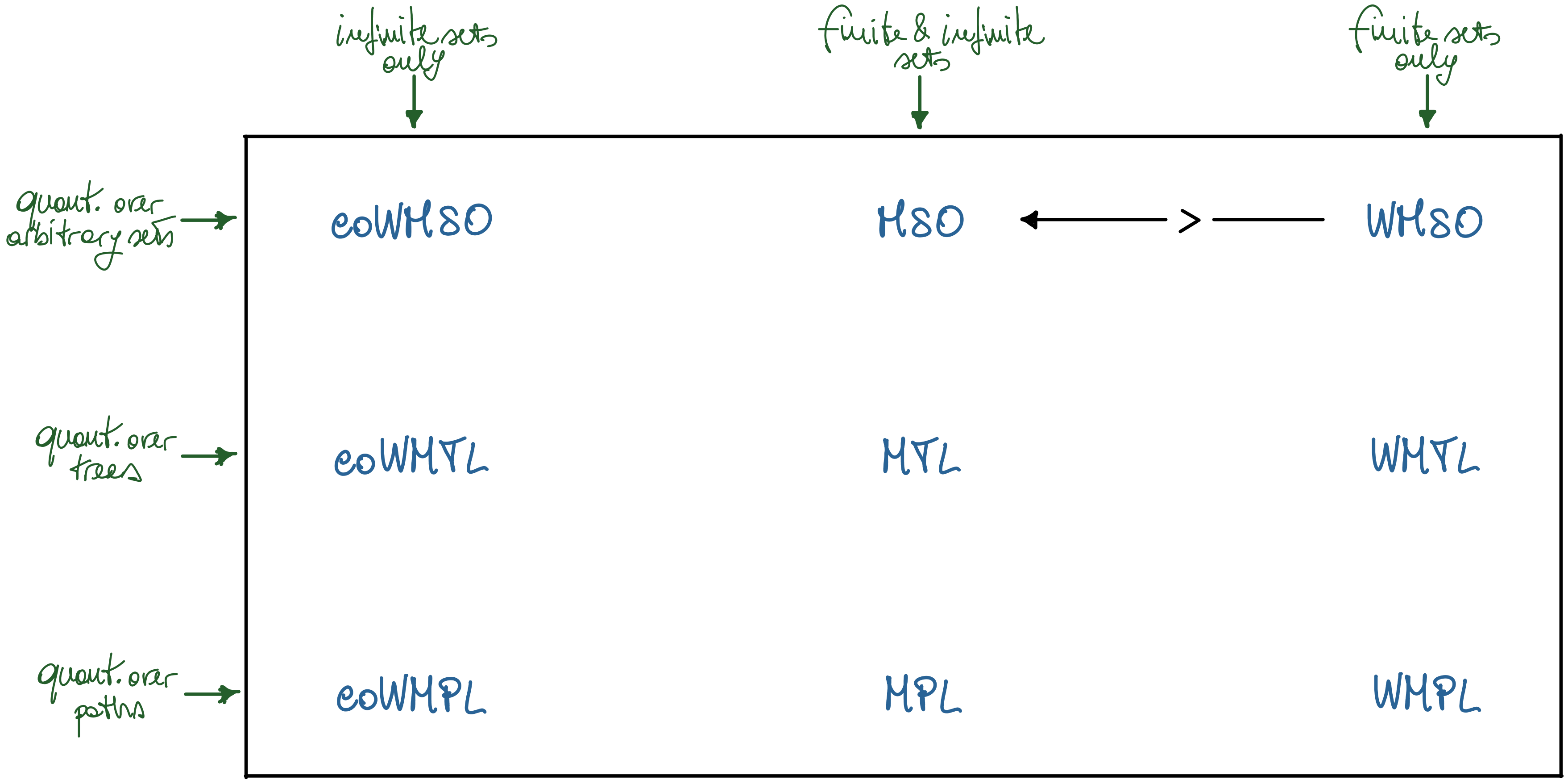
* Expressive comparisons with temporal logics

  → Introduction of a new non-trivial fragment of μCalculus subsumed by MTL

  → Proof of subsumption of well-known temporal & strategic logics

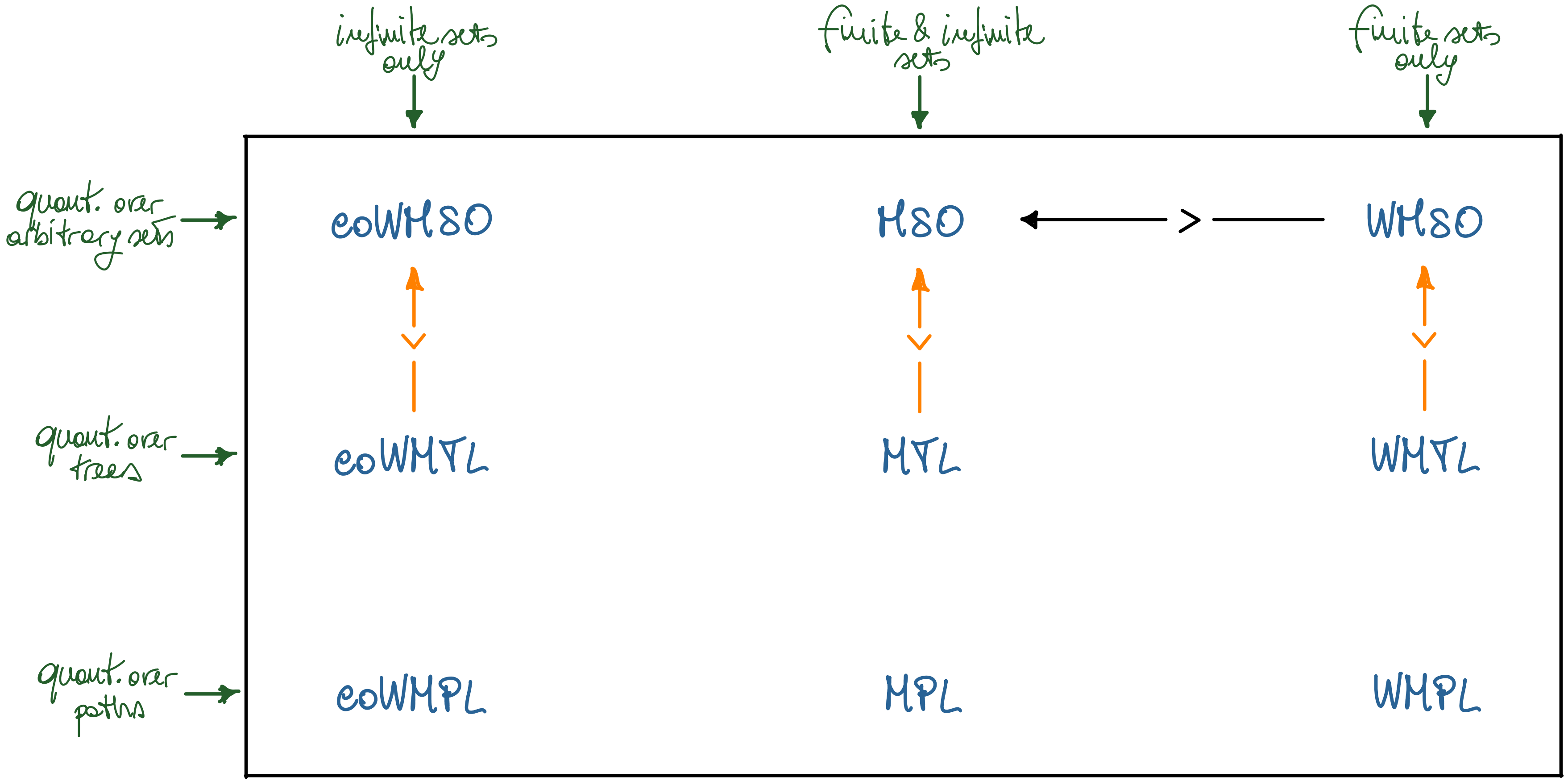
Expressiveness of MSOL Fragments

A Hierarchy of Fragments (On Finite-Branching Trees)



	infinite sets only	finite & infinite sets	finite sets only
quant. over arbitrary sets	coWMSO	MSO	WMSO
quant. over trees	coWMTL	MTL	WMTL
quant. over paths	coWMPL	MPL	WMPL

A Hierarchy of Fragments (On Finite-Branching Trees)



Separation Properties

Non-Connected Dependent Sets

e.g., AG_2P

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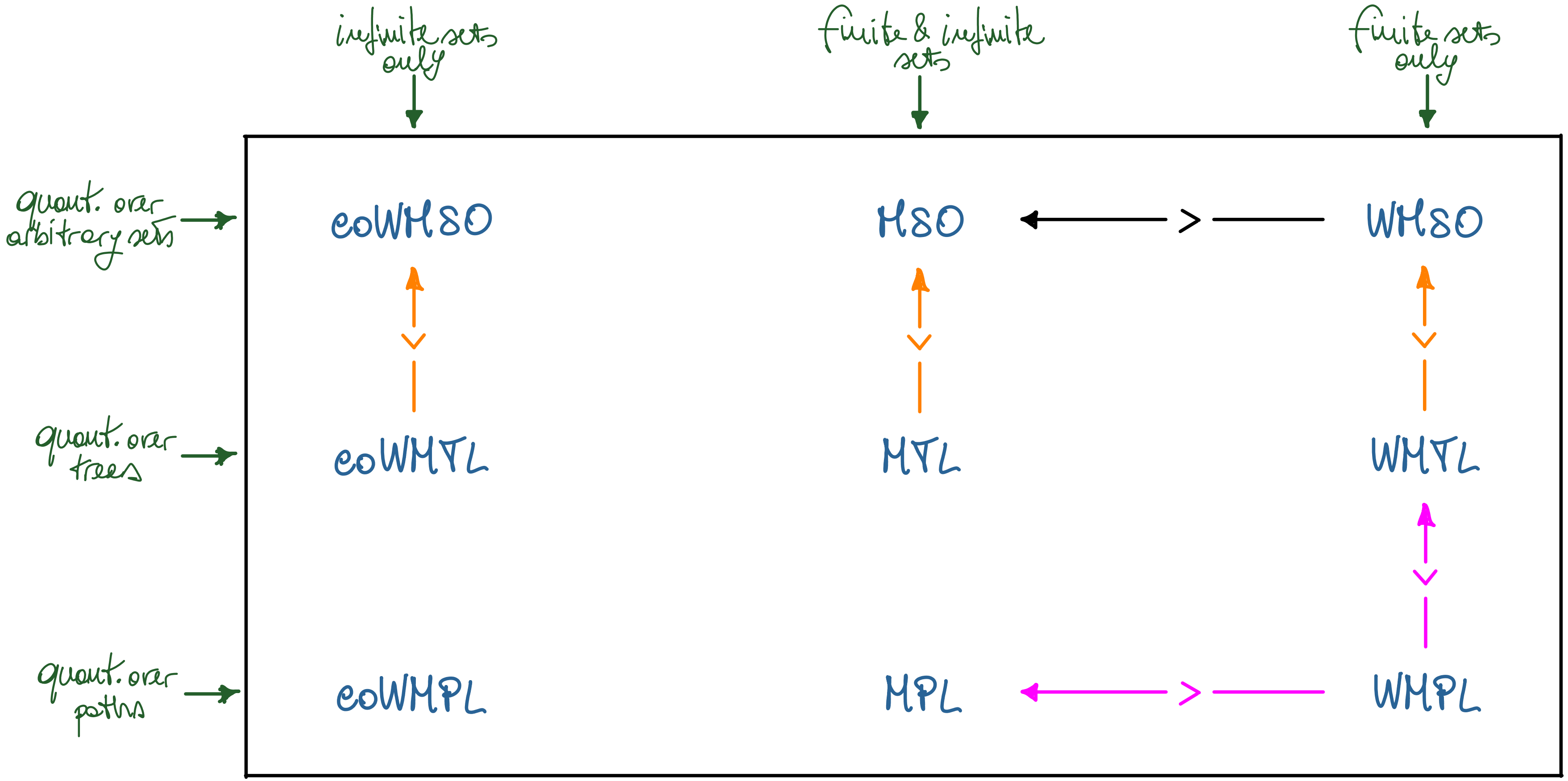

Separation Properties

↕ Non-Connected Dependent Sets e.g., AG_2P

↕ Unbounded Trees e.g., AFP

For ↕ we used the equiv.

- WMPL ≡ Counting-WCTL*

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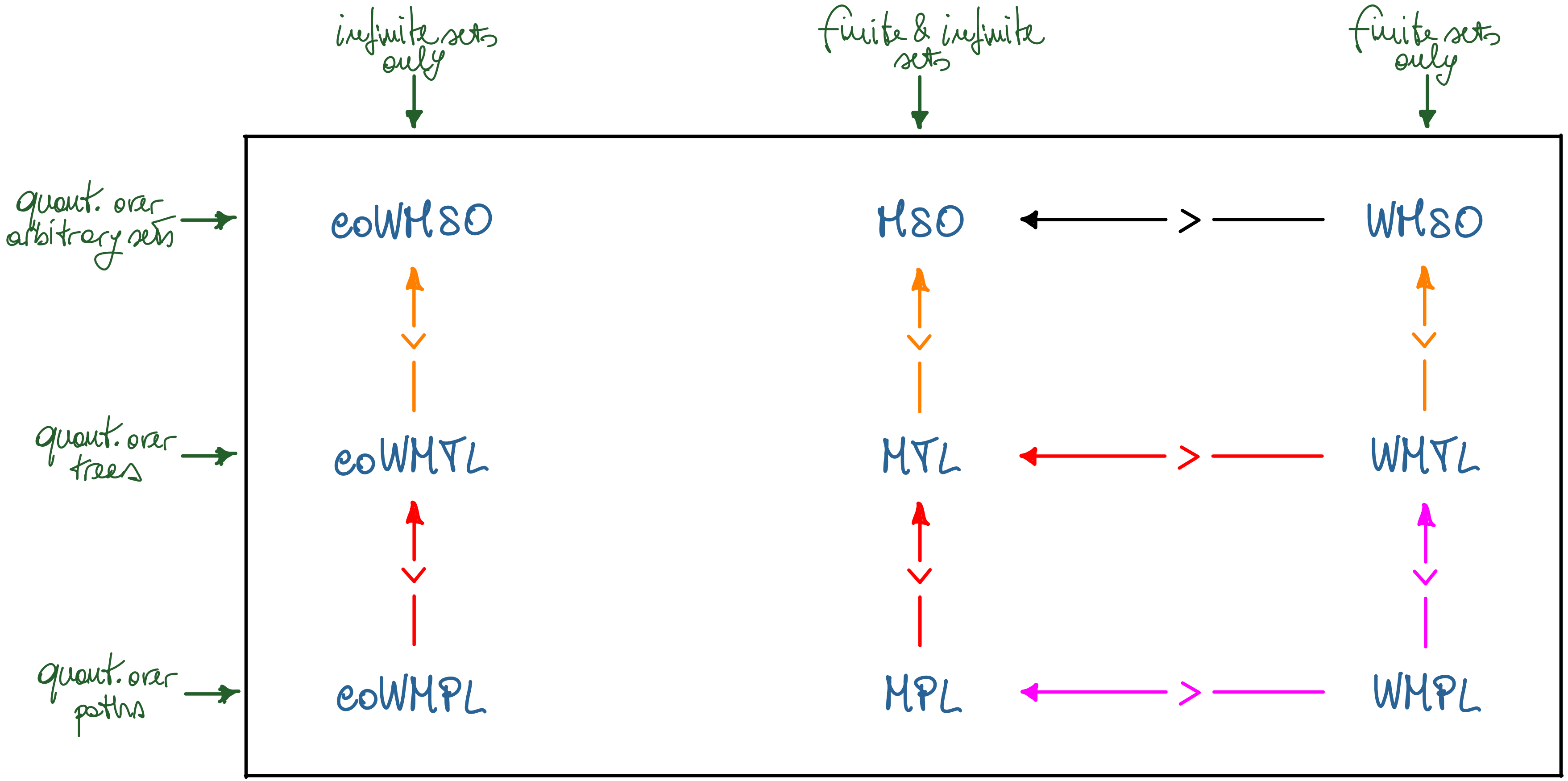

Separation Properties

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↕ Density of Subtrees
(SEE LATER)

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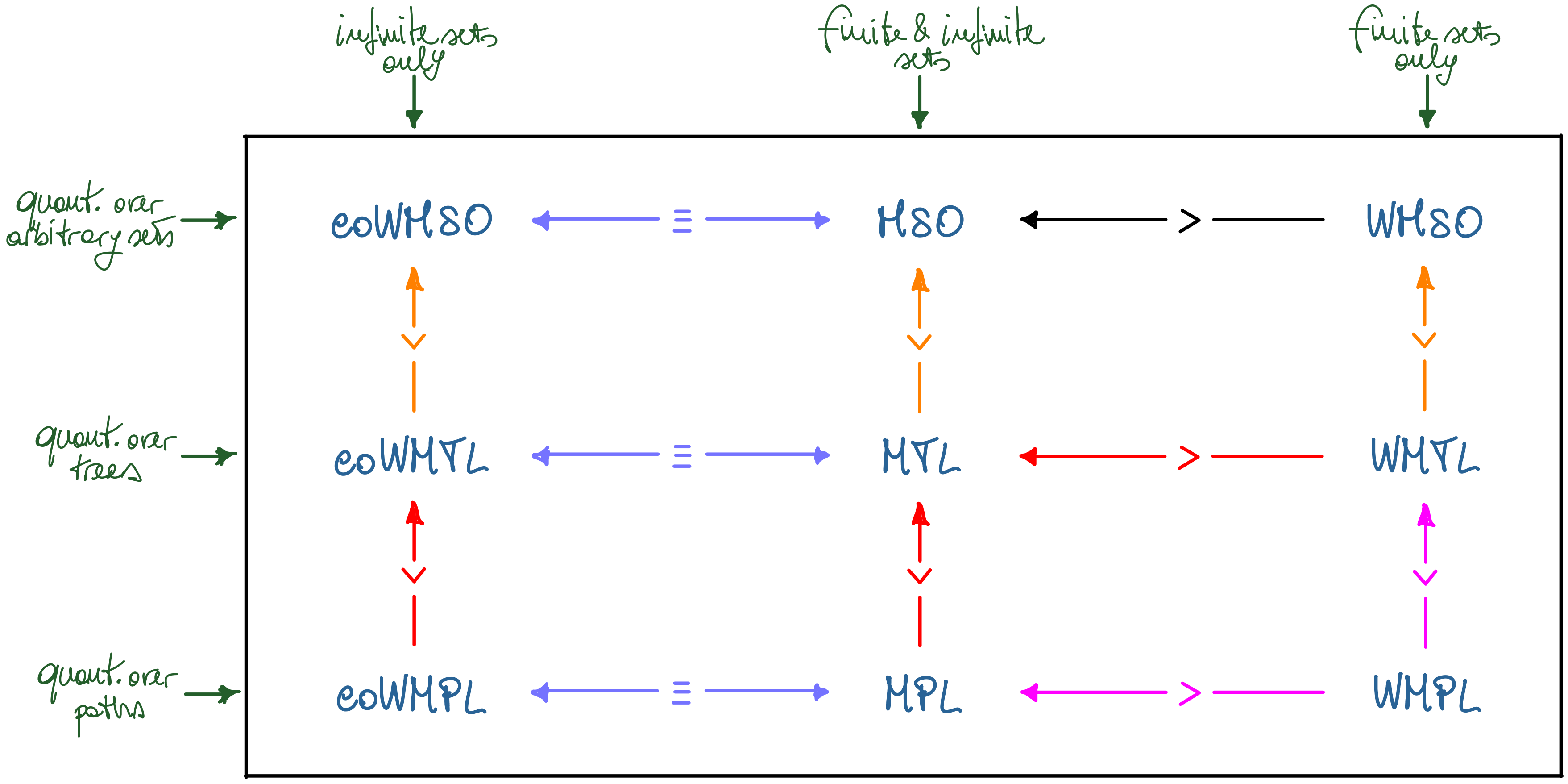

Separation Properties

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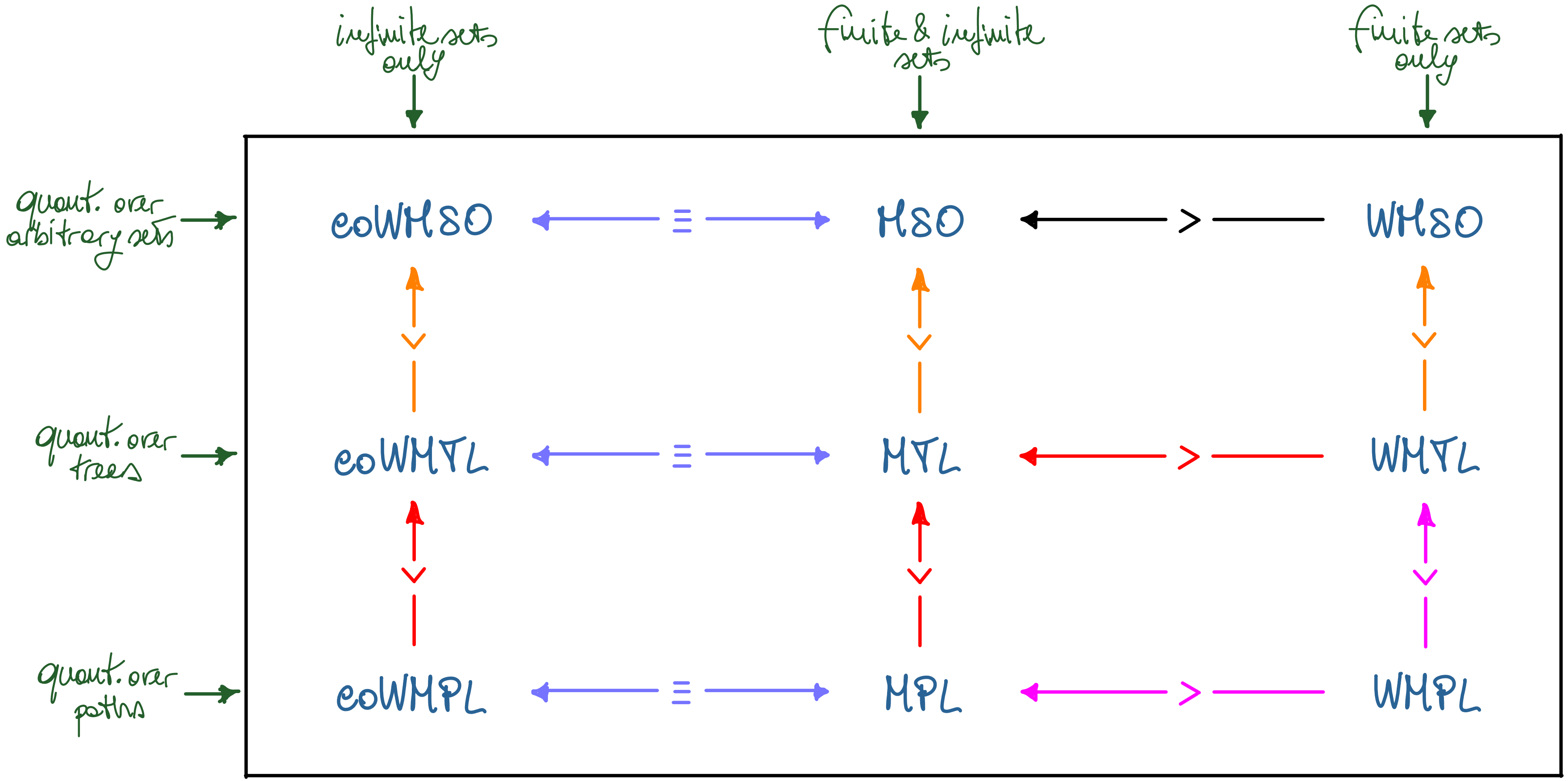
A HIERARCHY OF FRAGMENTS (ON FINITE-BRANCHING TREES)



SEPARATION PROPERTIES

↑ Non-Connected Dependent Sets e.g., $AG_2 P$

↑ Unbounded Trees e.g., $AF p$

↑ Density of Subtrees (SEE LATER)

For ↑ and ↑ we used the equiv.

- WMPL ≡ Counting-WCTL*
- MPL ≡ Counting-CTL*

- ⊆ : Infiniteness via Non-Wellfoundedness ← This breaks on infinite-branching trees
- ⊇ : Finiteness via FOL encoding

CoWeak MTL/MSO are more expressive than MTL/MSO in general

A Dense Family of Trees

* A tree T is Thick if it contains a binary tree as a minor (the subtree order is (quasi) dense)

- → MTL can easily express the density property
- → MPL ($\equiv$ Counting-CTL*) & WMSO ($\equiv_{Bis}$ AF-μCalculus) are unable to express such a property

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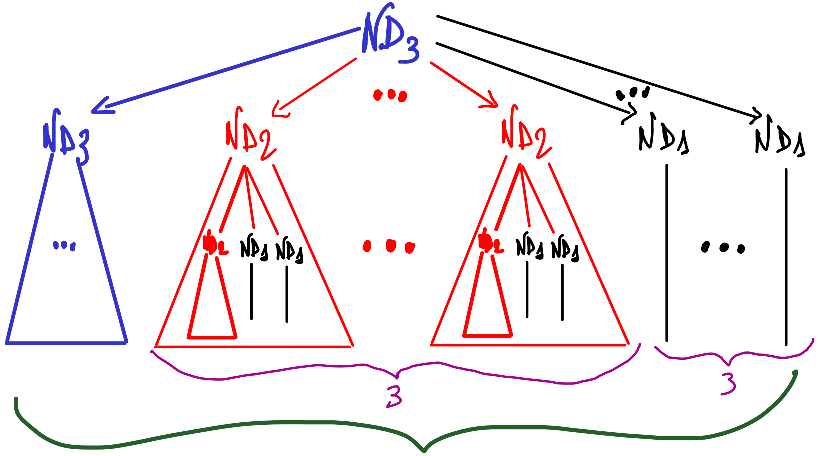

ND_3: A THIN TREE

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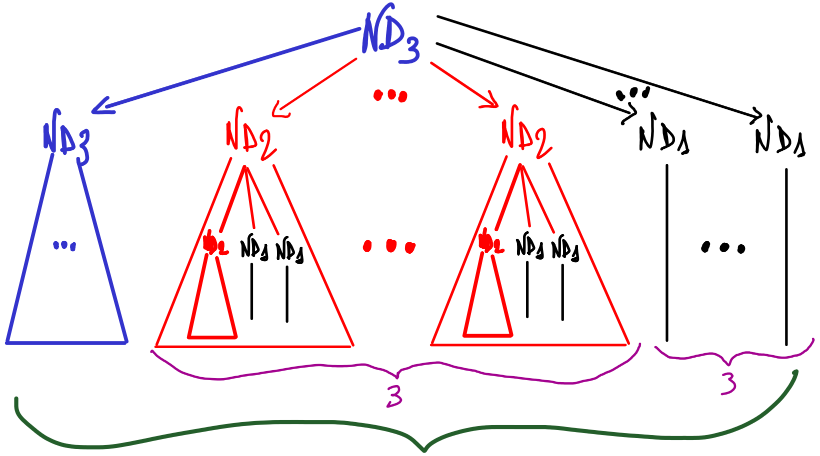
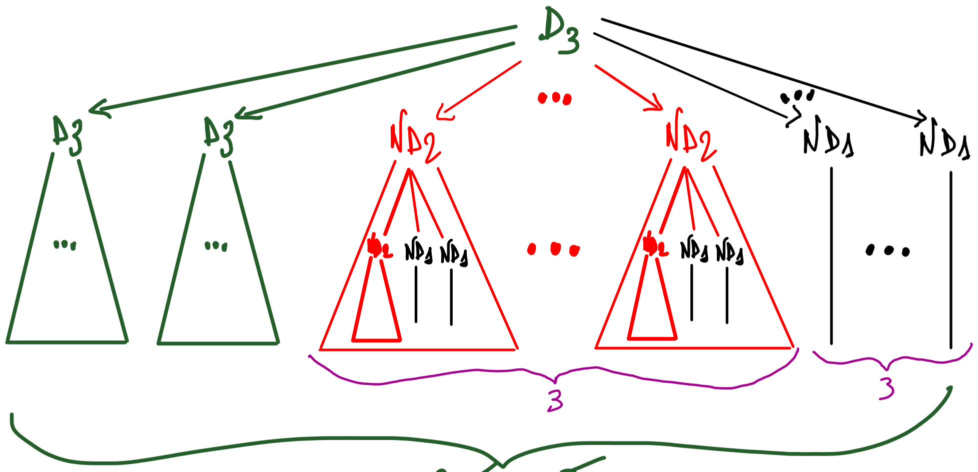


ND3: A Thin Tree



D3: A Thick Tree

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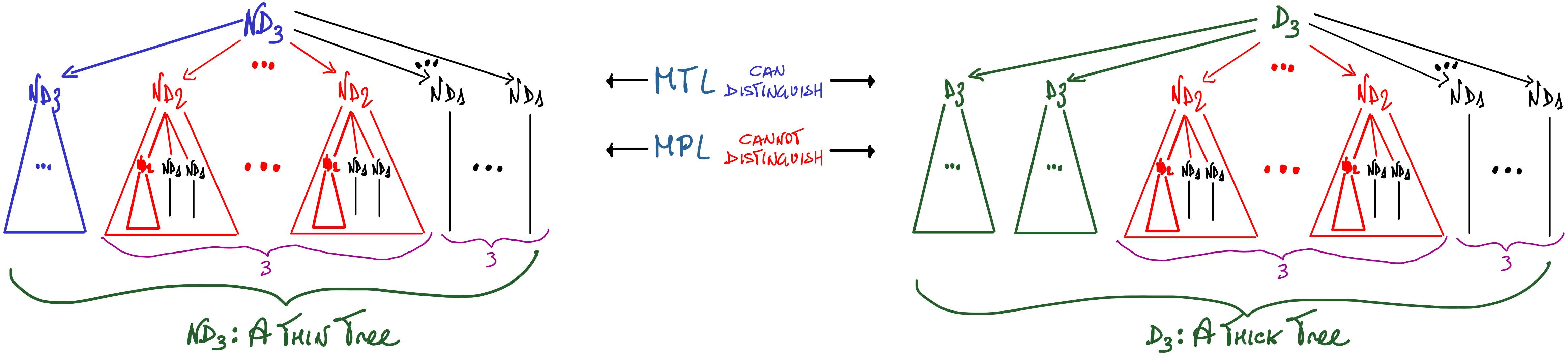

MSOL & Temporal Logics

One-Step Fragment of Graded-μ Calculus

$$\begin{tikzcd}
\text{G}\mu\text{Calculus} \arrow[r, leftrightarrow, "\equiv"] & \text{MSOL} \arrow[d, leftrightarrow] \\
 & \text{MTL} \arrow[d, leftrightarrow] \\
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$$\left\{\begin{aligned}
\vartheta_{Z,O} &::= \mu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \nu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \varphi_{Z,O} \\
\varphi_{Z,O} &::= p \mid X(\in Z) \mid \neg\varphi_{Z,O} \mid \varphi_{Z,O}\wedge\varphi_{Z,O} \mid \Diamond_{\geqslant k}\varphi_{O,\emptyset} \mid \vartheta_{\emptyset,\emptyset}
\end{aligned}\right.$$

One-Step Fragment of Graded-μ Calculus

$$\begin{tikzcd}
G\mu Calculus \arrow[r, leftrightarrow, "\equiv"] & MSOL \arrow[d, leftrightarrow, dashed] \\
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Zero-Step Variables

One-Step Variables

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Introduction of a new zero/one-step variable

$$\vartheta_{Z,O} ::= \mu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \nu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \varphi_{Z,O}$$

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One-Step Variables

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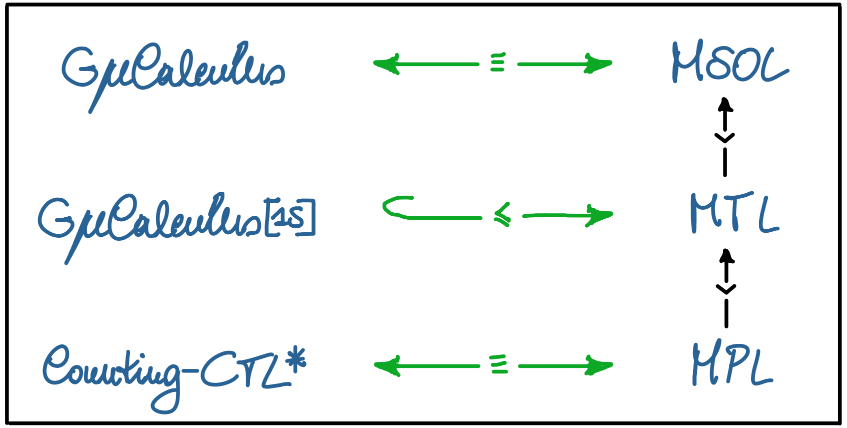

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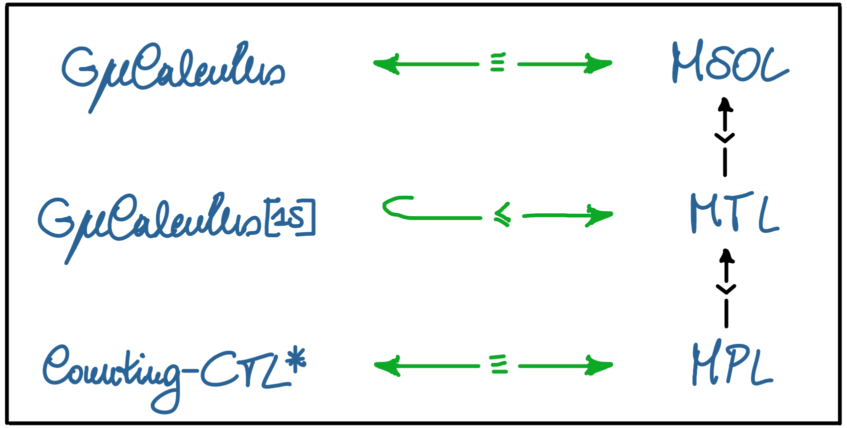
One-Step Variables

The matrix of a fixpoint formula has only free variables in $Z \cup O$

One-Step Fragment of Graded-μ Calculus



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\end{cases}
$$

The matrix of a fixpoint formula has only free variables in Z∪O

Zero-Step Variables

One-Step Variables

Only zero-step variables can be referred to directly

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\\
\varphi_{Z,O} \coloneqq p \mid X (\in Z) \mid \neg\varphi_{Z,O} \mid \varphi_{Z,O} \wedge \varphi_{Z,O} \mid \Diamond_{\geqslant k}\varphi_{O,\emptyset} \mid \vartheta_{\emptyset,\emptyset}
\end{array}\right.$$

The matrix of a fixpoint formula has only free variables in $Z \cup O$

Zero-Step Variables

One-Step Variables

Only zero-step variables can be referred to directly

A temporal modality
- Transforms one-step vars in zero-step vars
- Reset the one-step vars

One-Step Fragment of Graded-μ Calculus

$$\begin{tikzcd}
G\mu\text{Calculus} \arrow[rr, leftrightarrow, "\equiv"] & & \text{MSOL} \arrow[d, leftrightarrow, dashed] \\
G\mu\text{Calculus}[15] \arrow[rr, hook, "\leq"] & & \text{MTL} \arrow[d, leftrightarrow, dashed] \\
\text{Counting-CTL}^{*} \arrow[rr, leftrightarrow, "\equiv"] & & \text{MPL}
\end{tikzcd}$$

Introduction of a new zero/one-step variable

$$\vartheta_{Z,O} ::= \mu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \nu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \varphi_{Z,O}$$

The matrix of a fixpoint formula has only free variables in $Z \cup O$

Zero-Step Variables

One-Step Variables

New fixpoint formulae replenish the sets of variables

$$\varphi_{Z,O} ::= p \mid X (\in Z) \mid \neg\varphi_{Z,O} \mid \varphi_{Z,O} \wedge \varphi_{Z,O} \mid \Diamond_{\geq k}\varphi_{O,\emptyset} \mid \vartheta_{\emptyset,\emptyset}$$

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One-Step Fragment of Graded-μ Calculus

$$\begin{tikzcd}
G\mu Calculus \arrow[r, leftrightarrow, "\equiv"] & MSOL \\
G\mu Calculus[15] \arrow[r, hook, "\leqslant"] & MTL \arrow[u, leftrightarrow, dashed] \\
Counting\text{-}CTL^{*} \arrow[r, leftrightarrow, "\equiv"] & MPL \arrow[u, leftrightarrow, dashed]
\end{tikzcd}$$

INTRODUCTION OF A NEW ZERO/ONE-STEP VARIABLE

$$\vartheta_{Z,O} ::= \mu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \nu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \varphi_{Z,O}$$

THE MATRIX OF A FIXPOINT FORMULA HAS ONLY FREE VARIABLES IN $Z \cup O$

ZERO-STEP VARIABLES

ONE-STEP VARIABLES

NEW FIXPOINT FORMULAE REPLENISH THE SETS OF VARIABLES

$$\varphi_{Z,O} ::= p \mid X (\in Z) \mid \neg\varphi_{Z,O} \mid \varphi_{Z,O} \wedge \varphi_{Z,O} \mid \Diamond_{\geqslant k}\, \varphi_{O,\emptyset} \mid \vartheta_{\emptyset,\emptyset}$$

ONLY ZERO-STEP VARIABLES CAN BE REFERRED TO DIRECTLY

A TEMPORAL MODALITY

- TRANSFORMS ONE-STEP VARS IN ZERO-STEP VARS
- RESET THE ONE-STEP VARS

THE ZERO-STEP VAR SET AT THIS POSITION CONTAINS X

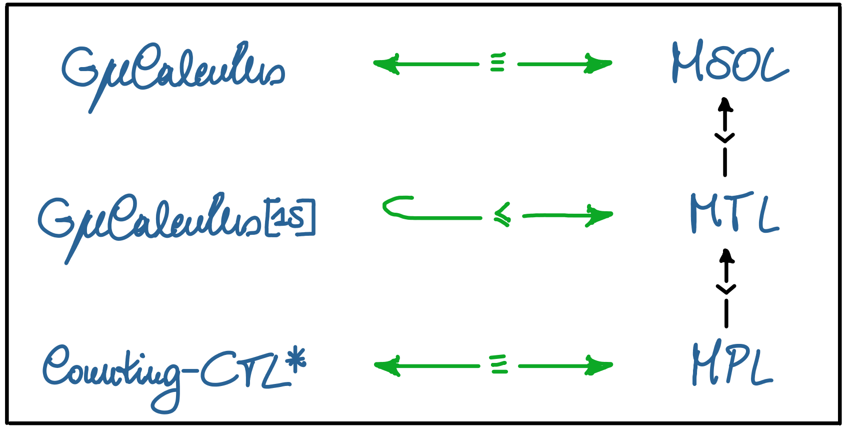
Examples:

$$\nu X.(p \wedge (q \vee \Diamond X)) \equiv E\, qRp \in G\mu Calculus[15]$$

$$\nu X.(p \wedge (q \vee \Diamond\Diamond X)) \equiv E\, qR_2 p \notin G\mu Calculus[15]$$

THE ZERO-STEP VAR SET AT THIS POSITION IS EMPTY

One-Step Fragment of Graded-μ Calculus



Introduction of a new zero/one-step variable

$$\vartheta_{Z,O} ::= \mu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \nu X.\vartheta_{Z\cup\{X\},O\cup\{X\}} \mid \varphi_{Z,O}$$

The matrix of a fixpoint formula has only free variables in $Z \cup O$

Zero-Step Variables

One-Step Variables

New fixpoint formulae replenish the sets of variables

$$\varphi_{Z,O} ::= p \mid X (\in Z) \mid \neg\varphi_{Z,O} \mid \varphi_{Z,O} \wedge \varphi_{Z,O} \mid \Diamond_{\geq k}\varphi_{O,\emptyset} \mid \vartheta_{\emptyset,\emptyset}$$

Only zero-step variables can be referred to directly

A temporal modality
- Transforms one-step vars in zero-step vars
- Reset the one-step vars

Examples:

The zero-step var set at this position contains X

$$\nu X.(p \wedge (q \vee \Diamond X)) \equiv E\,q R\,p \in \text{GμCalculus[15]}$$

$$\nu X.(p \wedge (q \vee \Diamond\Diamond X)) \equiv E\,q R_2\,p \notin \text{GμCalculus[15]}$$

The zero-step var set at this position is empty

GμCalculus[15] can express the density property

$$\boxed{\varphi_{DEN} \triangleq \nu X.\mu Y.(\Diamond_{\geq 2} X) \vee (\Diamond_{\geq 1} Y)}$$

A Semantically Non-Trivial Encoding

enc_x : GμCalculus[15] $\rightarrow$ MTL

$\mathrm{enc}_x : \quad \nu X.\theta \quad \mapsto \exists^{T} X.\ x \in X \wedge \forall y \in X.\ \mathrm{enc}_y(\theta)$

$$\left.\right\} \quad x \in [\![\theta]\!]^{T}_{\chi} \iff T, x, \chi \models \mathrm{enc}_x(\theta)$$

The fixpoint variable is interpreted as a tree where all nodes satisfy the matrix θ

A Semantically Non-Trivial Encoding

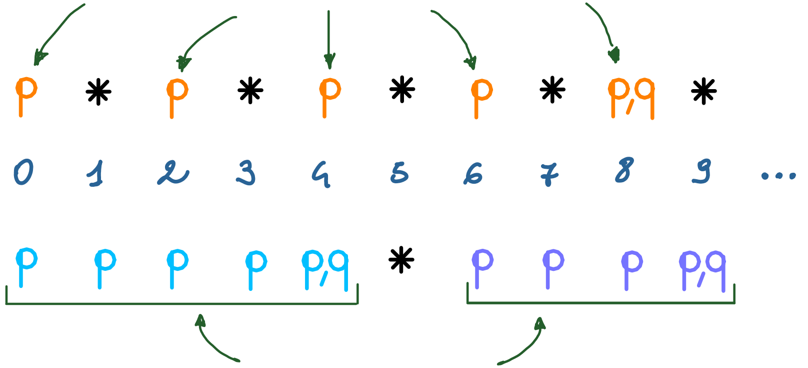
$$\left.\begin{array}{lll} enc_x : & G\mu Calculus[15] & \rightarrow MTL \\ enc_x : & \nu X.\vartheta & \mapsto \exists^T X.\, x \in X \wedge \forall y \in X.\, enc_y(\vartheta) \end{array}\right\} \; x \in [\![\vartheta]\!]^T_\chi \iff T, x, \chi \models enc_x(\vartheta)$$

The fixpoint variable is interpreted as a tree where all nodes satisfy the matrix ϑ

The connected components of the denotation do not satisfy the matrix $p \wedge (q \vee \Diamond\Diamond X)$ if taken in isolation

$$EqR_2 p \equiv \nu X.(p \wedge (q \vee \Diamond\Diamond X))$$

$$EqR p \equiv \nu X.(p \wedge (q \vee \Diamond X))$$



All connected components of the denotation satisfy the matrix $p \wedge (q \vee \Diamond X)$ if taken in isolation, i.e., they are self-sustainable

THANK YOU VERY MUCH