Quantifying over Trees in Henadic Second-Order Logics

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Monadic Second-Order Logic

* MSOL = FOL + Quantification over Sets of Elements
Homotopic Second-Order Logic

* HSOL = FO₂ + Quantification over Sets of Elements

- Undecidable over arbitrary structures
- Decidable on trees [Rabin '63]
- Decidable on graphs with bounded tree-width [Courcelle '83]
- Decidable on tree-like structures
**Monadic Second-Order Logic**

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- $\{\text{Sets of Computations viewed as Trees}\}$
  - Relation with Temporal Logics

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**References**

- [Rabin 1963](#)
- [Courcelle 1983](#)
Monadic Second-Order Logic

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→ Undecidable over arbitrary structures
→ Decidable on trees [Rabin 63]
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\{ Sets of Computations viewed as Trees \}
→ Relation with Temporal Logics

\[\mu\text{Calculus} \quad \rightarrow \quad \text{MSOL: Quantification over arbitrary sets of nodes}\]

\[\text{CTL}^* \quad \rightarrow \quad \text{HPL: Quantification over paths (connected linear orders)}\]
Monadic Second-Order Logic

* $\text{MSOL} = \text{FOL} + \text{Quantification over Sets of Elements}$

- Undecidable over arbitrary structures
- Decidable on trees \([\text{Rabin'63}]\)
- Decidable on graphs with bounded tree-width \([\text{Courcelle'89}]\)
- Decidable on tree-like structures

\[ \text{Sets of Computations viewed as Trees} \]
\[ \downarrow \]
\[ \text{Relation with Temporal Logics} \]

\[ \mu\text{Calculus} \]
\[ \text{MSOL: Quantification over arbitrary sets of nodes} \]

\[ \text{CTL*} \]
\[ \text{HPL: Quantification over paths (connected linear order)} \]
A Closer Look at the Embeddings

\[ (\exists q R_x \rho =) \quad \forall X.(\rho \land (q \lor \Box X)) \]

\[ (\forall X.(\rho \land (q \lor \Box X)) =) \quad \exists q R_x \rho \]

\[ \mu \text{calculus} \quad \overset{\sim}{\longrightarrow} \quad \text{HSOL} \]

\[ \text{CTL}^* \quad \overset{\sim}{\longrightarrow} \quad \text{HPL} \]
A Closer Look at the Embeddings

\[ \mu \text{Calculus} \rightarrow \text{MSOL} \]

\[ (\exists q R_2 p) \iff \exists X. (p \land (q \lor \diamond X)) \]

\[ \mu \text{Calculus} \rightarrow \text{CTL}^* \rightarrow \text{KPL} \]

\[ (\exists X. (p \land (q \lor \diamond X))) = \exists q R p \]

\[ 0 1 2 3 4 5 6 7 8 9 \ldots \]

\[ \exists q R p \]

\[ p \ p \ p \ p \ p q \ p q \ p q \]
A Closer Look at the Embeddings

μCalculus

(∀x. p(x) → ∃y. R(y, x))

(∃X. (p ∧ (q ∨ □X)))

(∀X. (p ∧ (q ∨ □X)))

∀X. (p ∧ (q ∨ □X))

μCalculus

HSOL

∀x ∈ X. p(x) ∧ (q(x) ∨ (∃y. R(y, x) ∧ R(x, y) ∧ ∃z ∈ X))

∀x ∈ X. p(x) ∧ (q(x) ∨ (∃y. R(y, x) ∧ ∃z ∈ X))

CTL*

HPL
A Closer Look at the Embeddings

\[ (\exists R) R p = \mathcal{L}_X (p \land (q \lor \diamond X)) \]

\[ (\forall X. (p \lor (q \land X))) = \exists R p \]

\[ \mu \text{Calculus} \]

\[ \mu \text{Calculus} \rightarrow \mu \text{SOL} \]

\[ \forall x \in X. p(x) \land \left( \exists y \exists z. R(x,y) \land R(y,z) \land z \in X \right) \]

\[ \forall x \in X. p(x) \land \left( \exists y. R(x,y) \land y \in X \right) \]

\[ \text{CTL}^* \rightarrow \text{KPL} \]
A Closer Look at the Embeddings

\[ (\varrho R, \varrho =) : \exists X. (p \land (q \lor \boxdot X)) \]

\[ (\exists X. (p \land (q \lor \boxdot X))) =: \varrho R \varrho p \]

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A Closer Look at the Embeddings

μ-calculus

\((\mathcal{E}qR \mathcal{P} =) \cup \mathcal{X}. (p \land (q \lor \diamond X))\)

\((\cup \mathcal{X}. (p \land (q \lor \diamond X)) =) \mathcal{E}q \mathcal{P}\)

CTL*

MSOL

\(\forall x \in X. p(x) \land (q(x) \lor \exists y. R(x,y) \land (\exists z. R(y,z) \land z \in X))\)

HPL
A Closer Look at the Embeddings

\[ (\forall x. (p \land (q \lor o \land x)) \implies p \land q) \]

\[ (\forall x. (p \land (q \lor o \land x)) \implies p \land q) \]

\[ (\forall x. (p \land (q \lor o \land x)) \implies p \land q) \]

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\[ (\forall x. (p \land (q \lor o \land x)) \implies p \land q) \]
A Closer Look at the Embeddings

μ-calculus

(\exists qR_2 p =) \forall X. (p \land (q \lor \Box X))

MSOL

(\forall X. (p \land (q \lor X)) =) \exists qR p

HPL

CTL*

\forall x e X. p(x) \land (q(x) \lor \exists y, z. R(x, y) \land R(y, z) \land z e X)

\forall x e X. p(x) \land (q(x) \lor \exists y, z. R(x, y) \land y e X)
A Closer Look at the Embeddings

\( \mu \mathcal{C} \text{alculus} \)

\[
(\exists q R_2 p =) \cup X. (p \land (q \lor \Diamond X))
\]

\[
(\exists X. (p \land (q \lor \Diamond X)) =) \quad \exists g R p
\]

\( \text{CTL}^* \)

\( \text{MSOL} \)

\[\forall x \in X. p(x) \lor \left( \exists y \exists z. R(x,y) \land R(y,z) \land x \neq X \right)\]

\( \text{HPL} \)
A Closer Look at the Embeddings

connected (i.e., non-convex) sets of dependent nodes

μCalculus

(\exists x_1 \ldots x_n \cdot p_1(x_1) \land \ldots \land p_n(x_n)) \rightarrow \forall x_1 \ldots x_n \cdot \neg p_1(x_1) \lor \ldots \lor \neg p_n(x_n)

MSOL

\forall x \cdot \exists y \cdot (p(x) \land (q(y) \lor \exists z \cdot r(z)))

HPL

\forall x \cdot \exists y \cdot (p(x) \land (q(y) \lor \exists z \cdot r(z)))

(\exists x \cdot (p \land (q \lor \exists x))) = \exists x \cdot (p \land (q \lor \exists x))
A Closer Look at the Embeddings

- Non-connected (i.e., non-closed) sets of dependent nodes
- Denotations of formulae are arbitrary sets of nodes

μ-Calculus

\[(\exists R.p \subseteq) \cup X.(p \land (q \lor \bot X))\]

Dependent elements

\[0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 7 \quad 8 \quad 9 \quad ...\]

\[(\forall X.(p \land (q \lor \bot X)) =) \quad \exists R.p\]

Dependent elements

\[\forall x \in X. p(x) \land (q(x) \lor (\exists y, z. R(x,y) \land R(y,z) \land z \in X))\]

ctl*

Connected (i.e., convex) set of dependent nodes

\{ Denotations of formulae are forests of independent trees \}

HPL

MSOL
* Connectedness of denotation holds for other logics as well: $\text{ATL}^*, \text{SL}, \text{STL}^*$, etc.

quantifications over strategies or substructures

boil down to quantifications over subtrees
A Missing Piece: Quantification over Trees

* Connectedness of denotation holds for other logics as well: $\mathit{ATL}^*$, $\mathit{SL}$, $\mathit{STL}^*$, etc.

  quantifications over strategies or substructures
  boil down to quantifications over subtrees

* A natural fragment: \textit{Monadic Tree Logic (MTL)} = $\mathit{MSOL}$ with quantifications over trees

  • An easy observation: MTL $<$ MSOL!

  • A harder question: Does quantification over trees reduce to quantification over paths, namely MPL $<$ MTL?
Our Contribution

* Expressive comparisons among fragments of $\text{HSOL}$
  → Introduction of new fragments of $\text{HSOL}$ called "co-Weak"
  → Analysis of interesting separation properties

* Expressive comparisons with temporal logics
  → Introduction of a new non-trivial fragment of logics subsumed by $\text{HTL}$
  → Proof of subsumption of well-known temporal & strategic logics
Expressions of HSOL Fragments
A Hierarchy of Fragments (On Finite-Banching Trees)

- Infinite sets only
- Finite & infinite sets
- Finite sets only

- Quant. over arbitrary sets
- Quant. over trees
- Quant. over paths

- eoWHSO
- HSO
- WHSO

- eoWHTL
- HTL
- WHTL

- eoWMPL
- MPL
- WMPL
A Hierarchy of Fragments (On Finite-Batching Trees)

Separation Properties
Non-Connected Dependence
\( e.g., \mathcal{A}(G,P) \)
A Hierarchy of Fragments (On Finite-Branched Trees)

Separation Properties
- Non-Connected Dependence
  - e.g., AGₚ
- Unbounded Trees
  - e.g., AFP

For our work, we used the equiv.: WHPL = Equivalent-WCTL⁺
A Hierarchy of Fragments (On Finite-Bounding Trees)

**Separation Properties**
- Non-Connected Dependence (e.g., AGxP)
- Unbounded Trees (e.g., AFP)
- Density of Subtrees (accessed)
- For and we used the equiv.
  - WHPL = Counting-WCTL*
A Hierarchy of Fragments (On Finite-Branched Trees)

Separation Properties
- Non-Connected Dependence
- Unbounded Trees
- Density of Subtrees

For $\uparrow$ and $\downarrow$, we used the equiv.
- $\text{WHPL} = \text{Counting-\text{CTL}^*}$
- $\text{HPL} = \text{Counting-\text{CTL}^*}$

- $\leq$: Infinities via Non-Well-foundedness
- $\preceq$: Infinities via FOL encoding
A Hierarchy of Fragments (On Finite-Branching Trees)

- \( \leq \): Finiteness via Non-Wellfoundedness
- \( \equiv \): Finiteness via FOL encoding

This breaks on infinite-branching trees

\( \text{eoWHSO} \leftrightarrow \equiv \rightarrow \text{HSO} \leftrightarrow > \rightarrow \text{WHSO} \)

\( \text{eoWHTL} \leftrightarrow \equiv \rightarrow \text{HTL} \leftrightarrow > \rightarrow \text{WHTL} \)

\( \text{eoWMPL} \leftrightarrow \equiv \rightarrow \text{MPL} \leftrightarrow > \rightarrow \text{WMPL} \)

Separation Properties
- Non-Connected Dependence, e.g., AG\(_2\)P
- Unbounded Trees, e.g., AFP
- Density of Subtrees (assumed)

For \( \uparrow \) and \( \downarrow \) we used the equiv.
- WHPL = Covering-WCTL\(_*\)
- HPL = Covering-CTL\(_*\)
A Dense Family of Trees

* A tree $T$ is *thick* if it contains a binary tree as a minor (the subtree order is (quasi) dense)

- MTL can easily express the density property
- MTL (Łoś's CT*) & WHSO (Łoś AF-molecule) are unable to express such a property
A Dense Family of Trees

* A tree $\tilde{T}$ is **thick** if it contains a binary tree as a minor (the subtree order is (quasi) dense).

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- MTL can easily express the density property.
- MPL (=Lowing-CT2*) & WHSO (=$\psi$-AF-molecular) are unable to express such a property.

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**ND$_3$: A Thin Tree**
A Dense Family of Trees

* A tree $T$ is thick if it contains a binary tree as a minor (the subtree order is (quasi) dense)

→ MTL can easily express the density property
→ MTL (≡ counting-$Ct^*$) & WHSO (≡ AF-$\mu$-calculus) are unable to express such a property

ND₃: A Thin Tree

D₃: A Thick Tree
A Dense Family of Trees

* A tree $T$ is **thick** if it contains a **binary tree as a mirror** (the subtree order is ( quasi) **degenerate**)

- **MTL** can easily express the density property
- **MPL** ($\equiv$ counting-CTC*) & **WHBO (E$_0$, AF-predicates)** are unable to express such a property

**ND$_3$: A Thin Tree**

**D$_3$: A Thick Tree**
HSQL & Temporal Logics
One-Step Fragment of Graded-μ Calculus

\[
\begin{align*}
\text{GμC} & \quad \equiv \quad \text{HSOL} \\
\text{Counting-CTL}^* & \quad \equiv \quad \text{HPL}
\end{align*}
\]
One-Step Fragment of Guarded-$\mu$ Calculus

\[ \theta_{z,0} := \mu X. \theta_{zutx,0utx} | \nu X. \theta_{zutx,0utx} | \varphi_{z,0} \]

\[ \varphi_{z,0} := p | X(z) | \neg \varphi_{z,0} | \varphi_{z,0} \land \varphi_{z,0} | \Box z \varphi_{0,0} | \varphi_{0,0} \]
One-Step Fragment of Graded-$\mu$ Calculus

\[
\begin{align*}
\Phi_{\mathcal{G}^\text{C}} & \iff \Phi_{\text{HPL}} \\
\Phi_{\mathcal{G}^\text{C}[\text{as}]} & \iff \Phi_{\text{MTL}} \\
\Phi_{\text{C-CTL}^*} & \iff \Phi_{\text{HPL}}
\end{align*}
\]

\[
\begin{align*}
\Theta_{z,0} & = \mu X. \Theta_{z(x),\text{out}(X)} | \nu X. \Theta_{z(x),\text{out}(X)} | \varphi_{z,0} \\
\varphi_{z,0} & = p | X(e_z) | \neg \varphi_{z,0} | \varphi_{z,0} \land \varphi_{z,0} | \exists_k \varphi_{0,0} | \theta_{0,0}
\end{align*}
\]

\text{Zero-Step Variables} \\
\text{One-Step Variables}
One-Step Fragment of Graded-\(\mu\) Calculus

\[\Theta_{z,i} := \mu X. \Theta_{z,\text{out}x_i} \mid \nu X. \Theta_{z,\text{out}x_i} \mid \Phi_{z,i}\]

- Zero-Step Variables
- One-Step Variables
One-Step Fragment of Graded-$\mu$ Calculus

Graphical representation:

- $\mu$Calculus $\equiv$ MSOL
- $\mu$Calculus,$[^{[35]}]$ $\equiv$ MTL
- Counting-CTL* $\equiv$ HPL

Formal expression:

$$\varphi_{Z,0} := \mu X. \varphi_{Z_{\text{ut}} X_1, \text{out} X_1} \mid \nu X. \varphi_{Z_{\text{ut}} X_1, \text{out} X_1} \mid \varphi_{Z,0}$$

The matrix of a fixpoint formula has only free variables in $Z_0$.

- Zero-Step Variables
- One-Step Variables
One-Step Fragment of Graded-\(\mu\)-Calculus

\[
\begin{align*}
\text{GmcCalculus} & \equiv \text{HSOL} \\
\text{GmcCalculus}^{[45]} & \equiv \text{MTL} \\
\text{Counting-CTL}^* & \equiv \text{HPL}
\end{align*}
\]

\[
\begin{align*}
\Theta_{Z,0} & := \mu X \cdot \Theta_{Z,0} \land \Theta_{X,0} \\
\Phi_{Z,0} & := \rho \cdot X(eZ) \land \Phi_{Z,0} \land \Phi_{Z,0} \land \Box_{Z} \Phi_{0,0} \land \Phi_{Z,0}
\end{align*}
\]

Introduction of a new zero/one-step variable.

The matrix of a fixed point formula has only free variables in \(Z\).

Only zero-step variables can be referred to directly.
Our-Step Fragment of Caesar+μCircuits

\[ \mathcal{L}_0 \equiv \mu X. \text{Success} \times X. \text{Success} \times \mathcal{L}_0 \]

Zero-Step Variables

\[ \mathcal{L}_0 \]

Directly

Only Zero-Step Variables

Can be reduced to

A Trajectory Portal

Reset the Ode-Step Was

\[ \text{The \ matrix \ of \ a \ portal \ formula} \]

Has only free variables in \( Z_0 \)
One-Step Fragment of Graded-μ Calculus

\[ \Phi_{Z,0} := \mu X. \Phi_{Z \cup X \cup \{y_X\}} \mid u X. \Phi_{Z \cup X \cup \{y_X\}} \mid \phi_{Z,0} \]

Introduction of a new zero/one-step variable

The matrix of a fixpoint formula has only free variables in \( Z \cup 0 \)

New fixpoint formulae replenish the set of variables

Zero-step variables

One-step variables

Only zero-step variables can be referred to directly

A temporal modality

Transforms one-step vars in zero-step vars

Reset the one-step vars

Examples:

\[ \forall X. (p \land (q \lor \neg X)) = \exists q R_p \in \text{GtcCalculus}^{[35]} \]

\[ \forall X. (p \land (q \lor \neg \neg X)) = \exists q R_4 p \not\in \text{GtcCalculus}^{[35]} \]

The zero-step var set at this position contains \( X \)

The zero-step var set at this position is empty
One-Step Fragment of Graded-$\mu$ Calculus

**Examples:**

\[ \forall X. (p \land (q \lor \Diamond X)) \equiv \mathcal{E} q X p \in \text{GrCalculus}^{[3]} \]

\[ \forall X. (p \land (q \lor \Diamond X)) \equiv \mathcal{E} q X p \notin \text{GrCalculus}^{[3]} \]

\[ \forall X. (p \land (q \lor \Diamond X)) \equiv \mathcal{E} q X p \notin \text{GrCalculus}^{[3]} \]

**The zero-step var set at this position contains X**

**The zero-step var set at this position is empty**

**The matrix of a fixpoint formula has only free variables in Z_0**

**New fixpoint formulae replenish the set of variables**

**Introduction of a new zero/one-step variable**

**New fixpoint formulae can express the density property**

\[ \forall X. \forall Y. (\Diamond X) \lor (\Diamond Y) \]
A Semantically Non-Trivial Encoding

\[ \text{eme}_\chi : \text{GnuCalculus}^{\chi} \rightarrow \text{MTL} \]

\[ \text{eme}_\chi : \forall X. \theta \quad \mapsto \quad \exists Y. \chi \land \forall y \in X. \text{eme}_\chi(\theta) \]

\[ \chi \in [\theta]_\chi \iff \widetilde{\chi}, \chi \vdash \text{eme}_\chi(\theta) \]

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The fixpoint variable is interpreted as a tree where all nodes satisfy the matrix \( \theta \)
A Semantically Non-Trivial Encoding

\[ \text{let } \xi : \text{GenCalculus}\rightarrow \text{MTL} \]

\[ \text{let } \lambda \xi \text{ s.t. } \forall X. \theta \rightarrow \exists X. \xi X \land \forall y \in X. \xi y \theta \]

The fixpoint variable is interpreted as a tree where all nodes satisfy the matrix \( \theta \).

\[ \text{EqR}_p = \forall X. (p \land (q \circ \circ \circ)) \]

The connected components of the denotation do not satisfy the matrix \( p \land (q \circ \circ \circ) \) if taken in isolation.

\[ \text{EqR}_p = \forall X. (p \land (q \circ \circ)) \]

All connected components of the denotation satisfy the matrix \( p \land (q \circ \circ) \) if taken in isolation, i.e., they are self-sustainable.
~••~ Thank you very much ~••~