Higher-dimensional subdiagram matching

Amar Hadzihasanovic [1 2] Diana Kessler [1]

[1]Tallinn University of Technology

[2]Quantinuum, 17 Beaumont Street
Oxford OX1 2NA, United Kingdom

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Introduction

Applications of higher-dimensional diagram rewriting:

- Formalisation of higher category theory, higher algebra, homotopy theory or combinatorial topology.

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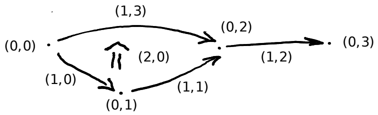
- Diagram rewriting as a model of computation.

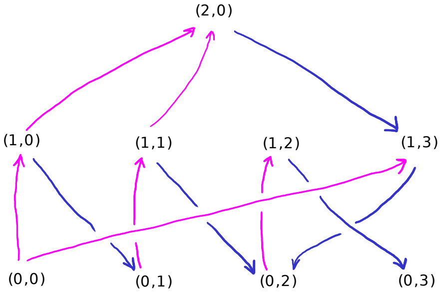
Goal:

Relate the derivational complexity of a higher-dimensional rewrite system to the worst-case time complexity of an implementation of a machine operating by higher-dimensional diagram rewriting.

Introduction

We encode the shape of a diagram by what we call an oriented graded poset.





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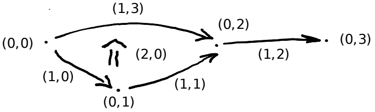

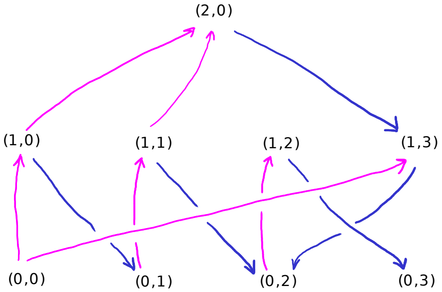

The well-formed shapes of diagrams form an inductive subclass of oriented graded posets called *regular molecules*.

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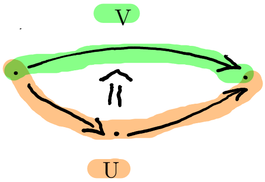

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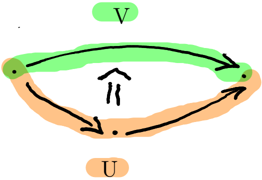
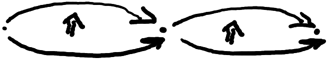
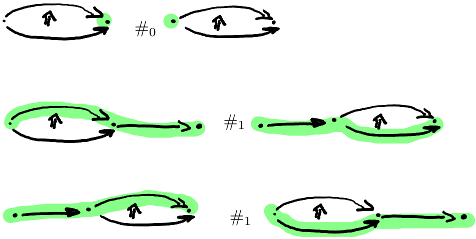


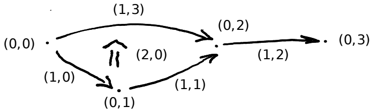
- (Paste): Let U, V be regular molecules satisfying some conditions. Then the pasting $U \#_k V$ is a regular molecule.

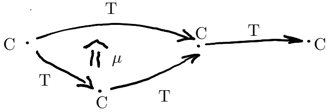




Diagrams

A diagram in our setting is a labelling of a regular molecule into a set of variables.



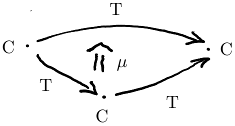


(0,0),(0,1),(0,2),(0,3) $\mapsto$ C

(1,0),(1,1),(1,2),(1,3) $\mapsto$ T

(2,0) $\mapsto$ μ

A cell is a diagram whose shape has one maximal element. This represents a basic rewrite.



Constructing regular molecules

Joint work with Amar Hadzihasanovic.

- Data structures for topologically sound higher-dimensional diagram rewriting.
  - ACT 2022, arXiv: 2209.09509.
  - Traversal algorithm for solving the isomorphism problem for regular molecules, running in time $O\left(n^2 \log n\right)$.
  - Already implemented in **rewalt** - https://rewalt.readthedocs.io.

Diagrammatic machines

A machine that operates by higher-dimensional rewriting works as follows:

- It has a list of $(n+1)$-dimensional rewrites whose input and output are n-dimensional diagrams.
- Given an n-dimensional diagram t as input, the machine tries to match one of the input boundaries of a rewrite rule to a rewritable subdiagram (portion) of t.
- If it finds such a match, then it replaces it with the output of the rewrite rule.

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Question: Is the obvious cost model that attributes constant cost to each rewrite step a "reasonable" cost model?

Higher-dimensional subdiagram matching

Given diagrams $s : V \to \mathbb{V}$ and $t \colon U \to \mathbb{V}$ such that $\dim(V) = \dim(U) = n$ and V is round, find if any all the inclusions $\imath \colon V \hookrightarrow U$ such that:

- $\imath(V) \sqsubseteq U$
  ($\imath(V)$ is a factor in a pasting decomposition of U),
- $s = \imath; t$
  (the labels match).

Higher-dimensional subdiagram matching

Given diagrams $s : V \to \mathbb{V}$ and $t : U \to \mathbb{V}$, the subdiagram matching problem can be split into three subproblems:

1. (Molecule matching problem) find, if any, the inclusions of V into U;
2. (Rewritable submolecule problem) decide if an inclusion is a *submolecule* inclusion;
3. check that the labelling is preserved.

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Problem 1 admits an algorithm that has a low-degree polynomial time bound in the sizes of U and V.

Problem 3 is trivial and should take linear time in the size of the diagram.

Problem 2 turns out to be more challenging.

Rewritable submolecule probelm

Theorem

The problem of deciding if an inclusion is a rewritable submolecule inclusion in dimension n can be solved in time

$$O\left(\prod_{k\leqslant n} |U_k|!\,|U_k|\right),$$

where U_k is the set of k-dimensional elements of U.

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Theorem

Every diagram of dimension $\leqslant 3$ is stably frame-acyclic.

Rewritable submolecule problem

Goal: Find if V is a factor in a pasting decomposition of U.

Problem: There are many possible decompositions. We need to "constrain" the space of possible decompositions.

The algorithm for arbitrary n: "Parametrise" this space by topological sorts of a DAG with at most $|U_n|$ number of vertices.

For stably frame-acyclic regular molecules: Either all topological sorts work or none works.

Future work

- Is there a PTIME algorithm for subdiagram matching in dimension 4?
- Is the problem of subdiagram matching in dimension 4 NP-complete?

Thank you!