Categorical account of composition methods in logic

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The setting

Let σ be a finite set of relational symbols with positive arities, we can define a category of σ-structures $\mathcal{R}(\sigma)$:

- Objects are $\mathcal{A} = (A, \{R^{\mathcal{A}}\}_{R\in\sigma})$ where $R^{\mathcal{A}} \subseteq A^r$ for r-ary relation symbol R.
- Morphisms $f : \mathcal{A} \to \mathcal{B}$ are relation preserving set functions $f : A \to B$

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Rightarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

- Embeddings $f \colon A \rightarrowtail B$ are injective functions which reflect relations:

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Leftarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

Model theory

Model theorists look at structures with the "blurry lens" of expressiblity in a logic $\mathcal{L}$:

$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \vDash \phi \Leftrightarrow \mathcal{B} \vDash \phi$$
$$\mathcal{A} \Rrightarrow^{\mathcal{L}} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \vDash \phi \Rightarrow \mathcal{B} \vDash \phi$$

Used to show inexpressiblity of properties in $\mathcal{L}$.

Typically the relations $\Rrightarrow^{\mathcal{L}}$ and $\equiv^{\mathcal{L}}$ are characterized by a model-comparision game, e.g. Ehrenfeucht-Fraïssé, pebble game, modal bisimulation game.

To mitigate difficulty in direct game-theoretic arguments for proving inexpressiblity higher-level techniques such as locality and *composition methods* are employed.

Composition Methods

Structures are built up recursively from applications of operations on a collection of basic structures.

For a logic $\mathcal{L}$, the operations H must must preserve $\equiv^{\mathcal{L}}$, i.e. must satisfy a Feferman-Vaught-Mostowski theorem:

Formulation

For an operation $H\colon \mathcal{C}_1 \times \mathcal{C}_2 \cdots \times \mathcal{C}_n \to \mathcal{D}$ *and logics* $\mathcal{J}_1, \ldots, \mathcal{J}_n, \mathcal{J}$*:*

$$\mathcal{A}_i \equiv^{\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv^{\mathcal{J}} H(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

with $\mathcal{A}_i, \mathcal{B}_i \in \mathcal{C}_i$ *where* $\mathcal{C}_i, \mathcal{D}$ *are relevant categories of models.*

Key ingredient in Courcelle's theorems and other algorithmic metatheorems.

A categorical approach

Some of the first such instances of FVM theorems were showing that $\equiv_{\mathbf{FOL}}$ is preserved by (co)products

Theorem (Feferman+Vaught 1959, Mostowski 1952)

If $\mathcal{A}_i \equiv_{\mathbf{FOL}} \mathcal{B}_i$ for all $i \in I$, then

- *Products:* $\mathcal{A}_1 \times \mathcal{A}_2 \equiv_{\mathbf{FOL}} \mathcal{B}_1 \times \mathcal{B}_2$ *and* $\prod_i \mathcal{A}_i \equiv_{\mathbf{FOL}} \prod_i \mathcal{B}_i$
- *Coproducts:* $\mathcal{A}_1 \uplus \mathcal{A}_2 \equiv_{\mathbf{FOL}} \mathcal{B}_1 \uplus \mathcal{B}_2$ *and* $\coprod_i \mathcal{A}_i \equiv_{\mathbf{FOL}} \coprod_i \mathcal{B}_i$

How can we prove such FVM theorems categorically?

Operations on models are functorial, logical equivalence is comonadic, and distributive laws witness their interaction.

Logical equivalence is comonadic

The game comonad framework has shown that a plethora of model comparison games can be encoded as comonads over categories of models

- Pebble games for finite variable logic (Abramsky+Dawar+Wang 17)
- Ehrenfeucht-Fraïssé games for first-order logic up to quantifier rank $\leq k$
- modal logic graded by depth
- guarded logics (Abramsky+Marsden 20)
- hybrid/bounded logics (Abramsky+Marsden 21)
- logics with generalised quantifiers (O'Conghaile+Dawar 20)
- logics with restricted conjunction (Montacute+S 22)

All of these are examples of arboreal covers which are studied axiomatically in Abramsky+Reggio 21

Running Example: Ehrenfeucht-Fraïssé comonad and $\uplus$

Given a σ-structure $\mathcal{A}$, we can create σ-structure $\mathbb{E}_k\mathcal{A}$ on non-empty sequences of elements in A of length $\leq k$

Let $\varepsilon_{\mathcal{A}} : \mathbb{E}_k\mathcal{A} \to \mathcal{A}$ return the last move of the play $[a_1, \ldots, a_n] \mapsto a_n$.

$$R^{\mathbb{E}_k\mathcal{A}}(s_1, \ldots, s_r) \Leftrightarrow \forall i, j \in [r],\ s_i \text{ is a prefix of } s_j \text{ or } s_j \text{ is a prefix of } s_i \\ \text{and } R^{\mathcal{A}}(\varepsilon_{\mathcal{A}}(s_1), \ldots, \varepsilon_{\mathcal{A}}(s_r))$$

Comultiplication $\delta \colon \mathbb{E}_k\mathcal{A} \to \mathbb{E}_k\mathbb{E}_k\mathcal{A}$ where

$$\delta([a_1, \ldots, a_n]) = [[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_n]]$$

Universe of $\mathcal{A}_1 \uplus \mathcal{A}_2 = \{(i, a_i) \mid i = \{1, 2\}, a_i \in A_i\}$ and obvious relations:

$$R^{\mathcal{A}_1 \uplus \mathcal{A}_2}((i_1, a_1), \ldots, (i_r, a_r)) \Leftrightarrow \exists i \in \{1, 2\} \forall j \in [r], i_j = i \\ \text{and } R^{\mathcal{A}_i}(a_1, \ldots, a_r)$$

Kleisli category $\mathbf{Kl}(\mathbb{E}_k)$ for $\mathbb{E}_k$,

- Objects sames as $\mathcal{R}(\sigma)$,
- Morphisms of type $\mathcal{A} \to \mathcal{B} \in \mathbf{Kl}(\mathbb{E}_k)$ are morphisms of type $\mathbb{E}_k A \to B \in \mathcal{R}(\sigma)$,
- Composition $g \cdot f = g \circ \mathbb{E}_k(f) \circ \delta_{\mathcal{A}}$, and identity $\varepsilon_A \colon \mathbb{E}_k \mathcal{A} \to \mathcal{A}$

Will use $\mathcal{A} \Rrightarrow_{\exists^+\mathbb{E}_k} \mathcal{B}$ to denote there exists a Kleisli morphism $f \colon \mathcal{A} \to_{\mathbb{E}_k} \mathcal{B}$

Theorem

- $\mathcal{A} \Rrightarrow_{\exists^+\mathbb{E}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \Rrightarrow_{\exists^+\mathbf{FOL}_k} \mathcal{B}$

$\exists^+\mathbf{FOL}_k$ is positive existential first-order logic up to rank $\leq k$

Observe that for every $\mathcal{A}_1, \mathcal{A}_2$ there is a morphism $\kappa_{\mathcal{A}_1,\mathcal{A}_2} \colon \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \to \mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2$

$$\kappa([(i_1, a_1), \ldots, (i_n, a_1)]) = \begin{cases} [a_j \mid i_j = 1] & \text{if } i_n = 1 \\ [a_j \mid i_j = 2] & \text{if } i_n = 2 \end{cases}$$

If $\mathcal{A}_i \Rrightarrow_{\exists^+\mathbb{E}_k} \mathcal{B}_i$ for $i \in \{1, 2\}$, then $f_i \colon \mathbb{E}_k\mathcal{A}_i \to \mathcal{B}_i$

$$\mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \xrightarrow{\kappa_{\mathcal{A}_1,\mathcal{A}_2}} \mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2 \xrightarrow{f_1 \uplus f_2} \mathcal{B}_1 \uplus \mathcal{B}_2$$

So $\mathcal{A}_1 \uplus \mathcal{A}_2 \Rrightarrow_{\exists^+\mathbb{E}_k} \mathcal{B}_1 \uplus \mathcal{B}_2$ and $\mathcal{A}_1 \uplus \mathcal{A}_2 \Rrightarrow_{\exists^+\mathbf{FOL}_k} \mathcal{B}_1 \uplus \mathcal{B}_2$

Theorem (Abstract $\exists^+$-FVM Theorem)

Given

- *operation* $H\colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$,
- *for all* $\mathcal{A}_1 \in \mathcal{C}_1, \ldots, \mathcal{A}_n \in \mathcal{C}_n$,
  $\kappa_{\mathcal{A}_1,\ldots,\mathcal{A}_n}\colon \mathbb{D}(H(\mathcal{A}_1,\ldots,\mathcal{A}_n)) \to H(\mathbb{C}_1(\mathcal{A}_1),\ldots,\mathbb{C}_n(\mathcal{A}_n))$

Then:

$$\mathcal{A}_i \Rrightarrow_{\exists^+\mathbb{C}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1,\ldots,\mathcal{A}_n) \Rrightarrow_{\exists^+\mathbb{D}} H(\mathcal{B}_1,\ldots,\mathcal{B}_n)$$

For the comonad $\mathbb{E}_k$, the morphism $\kappa_{\mathcal{A}_1,\ldots,\mathcal{A}_n}$ decomposes a Spoiler strategy on $H(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ in the one-sided EF-game to the individual Spoiler strategies on the component structures $\mathcal{A}_1,\ldots,\mathcal{A}_n$.

If this translation $\kappa_{\mathcal{A}_1,\ldots,\mathcal{A}_n}$ satisfies additional axioms, it is also a way of talking about strategies in the ordinary EF game and bijective EF game.

$\mathcal{A} \equiv_{\#\mathbb{E}_k} \mathcal{B}$ means there exists an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{Kl}(\mathbb{E}_k)$

Theorem

- $\mathcal{A} \equiv_{\#\mathbb{E}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv_{\#\mathbf{FOL}_k} \mathcal{B}$

If $f_i \colon \mathbb{E}_k \mathcal{A}_i \to \mathcal{B}_i$ is an isomorphism in $\mathbf{Kl}(\mathbb{E}_k)$ for $i \in \{1, 2\}$, then $f_1 \uplus f_2 \circ \kappa$ is an isomorphism. Follows from κ being Kleisli law

$$\varepsilon_{\mathcal{A}_1} \uplus \varepsilon_{\mathcal{A}_2} = \kappa \circ \varepsilon_{\mathcal{A}_1 \uplus \mathcal{A}_2} \quad \delta_{\mathcal{A}_1} \uplus \delta_{\mathcal{A}_2} \circ \kappa = \kappa \circ \mathbb{E}_k \kappa \circ \delta_{\mathcal{A}_1 \uplus \mathcal{A}_2}$$

Theorem (Abstract #-FVM theorem)

Given

- *operation* $H \colon C_1 \times \cdots \times C_n \to D$,
- *Kleisli law* $\kappa \colon \mathbb{D}(H(\mathcal{A}_1, \ldots, \mathcal{A}_n)) \to H(\mathbb{C}_1(\mathcal{A}_1), \ldots, \mathbb{C}_n(\mathcal{A}_n))$

Then:

$$\mathcal{A}_i \equiv_{\#\mathbb{C}_i} \mathcal{B}_i \textit{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv_{\#\mathbb{D}} H(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

Abstractly, objects of $\mathbf{EM}(\mathbb{E}_k)$ are morphisms of the form $\alpha\colon \mathcal{A} \to \mathbb{E}_k\mathcal{A}$.

Cofree coalgebra functor $G^{\mathbb{E}_k}\colon \mathcal{R}(\sigma) \to \mathbf{EM}(\mathbb{E}_k)$ where $\mathcal{A} \mapsto (\mathbb{E}_k\mathcal{A}, \delta_{\mathcal{A}})$

Concretely, $\mathbf{EM}(\mathbb{E}_k)$ represent forest-shaped covers of objects in $\mathcal{R}(\sigma)$ of height $\leq k$. Subcategory of paths $(P,\pi) \in \mathcal{P} \subseteq \mathbf{EM}(\mathbb{E}_k)$

$\mathcal{A} \equiv_{\mathbb{E}_k} \mathcal{B}$ if there exists a span $G^{\mathbb{E}_k}(\mathcal{A}) \xleftarrow{f} (X,\chi) \xrightarrow{g} G^{\mathbb{E}_k}(\mathcal{B})$ in $\mathbf{EM}(\mathbb{E}_k)$ where f, g are

- Pathwise embeddings $e\colon (P,\pi) \rightarrowtail (X,\chi)$ implies $f \circ e\colon (P,\pi) \rightarrowtail G^{\mathbb{E}_k}(\mathcal{A})$
- Open maps, a path which can be extended in the codomain, can be extended in the domain.

Theorem

$$\mathcal{A} \equiv_{\mathbb{E}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv_{\mathbf{FOL}_k} \mathcal{B}$$

where $\mathcal{L}_k$ is first-order logic up to quantifier rank $\leq k$ without equality.

Theorem (Abstract FVM theorem)

Given

- *operation H that preserves embeddings,*
- *Kleisli law $\kappa \colon \mathbb{D}(H(\mathcal{A}_1, \ldots, \mathcal{A}_n)) \to H(\mathbb{C}_1(\mathcal{A}_1), \ldots, \mathbb{C}_n(\mathcal{A}_n))$ satisfying the axiom:*
  - *for all paths $(P, \pi) \in \mathbf{EM}(\mathbb{D})$ and coalgebras $(\mathcal{A}_i, \alpha_i) \in \mathbf{EM}(\mathbb{C}_i)$ where*

$$
\begin{array}{ccccc}
P & & \xrightarrow{\quad f \quad} & & H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \\
\downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle H(\alpha_1, \ldots, \alpha_n)} \\
\mathbb{D}(P) & \xrightarrow{\mathbb{D}(f)} & \mathbb{D}(H(\mathcal{A}_1, \ldots, \mathcal{A}_n)) & \xrightarrow{\kappa} & H(\mathbb{C}_1(\mathcal{A}_1), \ldots, \mathbb{C}_n(\mathcal{A}_n))
\end{array}
\tag{1}
$$

commutes, f has 'minimal' decomposition as $f = H(e_1, \ldots, e_n) \circ e_0$ where $e_i \colon (P_i, \pi_i) \rightarrowtail (\mathcal{A}_i, \alpha_i) \in \mathbf{EM}(\mathbb{C}_i)$

Then:

$$\mathcal{A}_i \equiv_{\mathbb{C}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv_{\mathbb{D}} H(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

First stated on via a lifted operation $\hat{H}$ on coalgebras. Dualising ideas from commutative monad theory allowed us to restate the axioms in terms of H.

Concluding remarks

- The abstract FVM theorems with $H = \mathbf{Id}$ demonstrate logic of $\mathbb{D}$ is a fragment of the logic of $\mathbb{C}$.
- Using relative comonads, we obtain an abstract FVM theorem for logics with intepreted symbols, e.g. equality, connectivity predicate, second-order logics.
- Logics captured by an arboreal cover have an FVM theorem over the categorical product.
- Future work to address the more general form of FVM theorem where the indexing set is a structure itself.