Revisiting Membership Problems in Subclasses of Rational Relations

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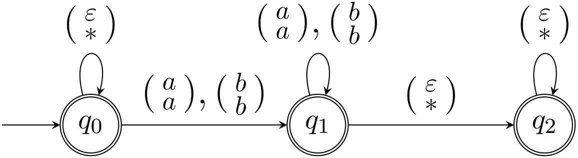
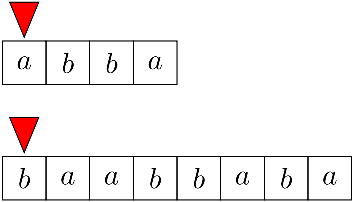
Rational Relations

$R = \{(x, y) \mid x \text{ is an infix of } y\}$





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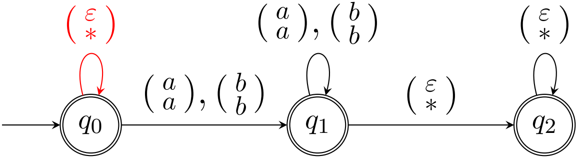

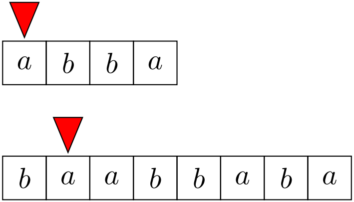

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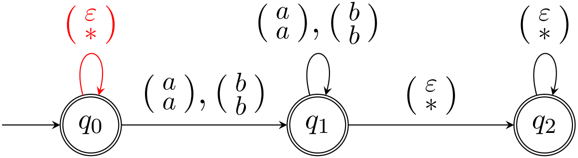

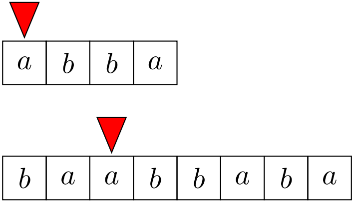

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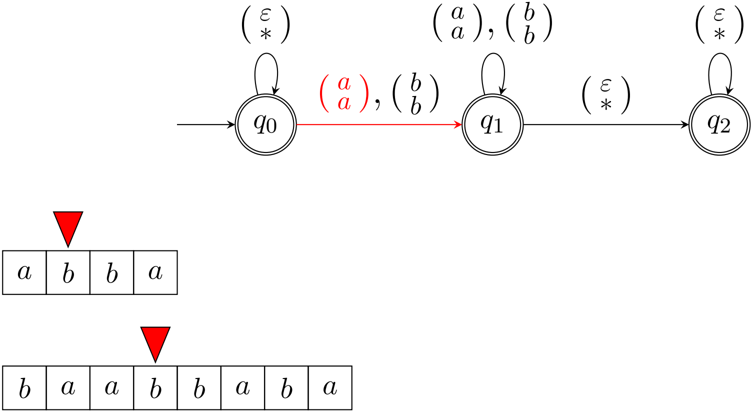

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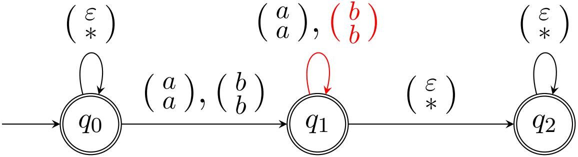

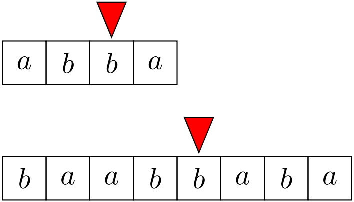

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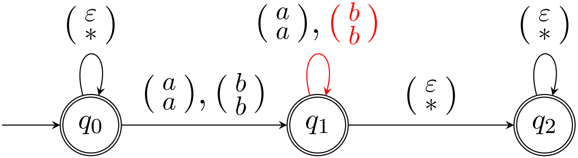

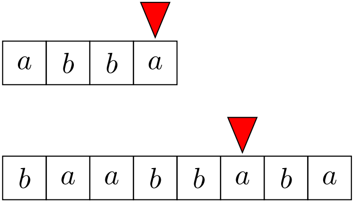

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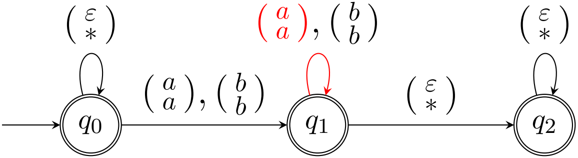

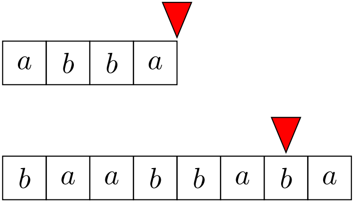

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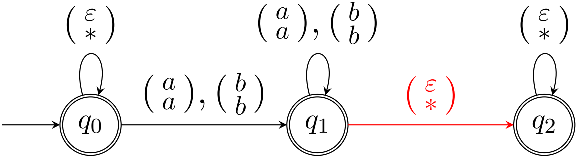

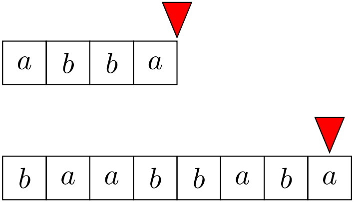

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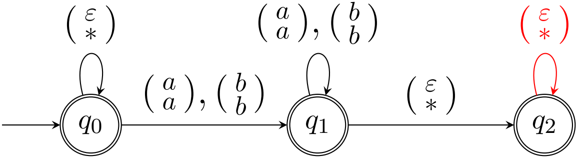
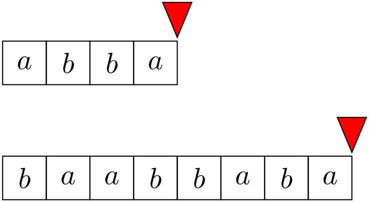
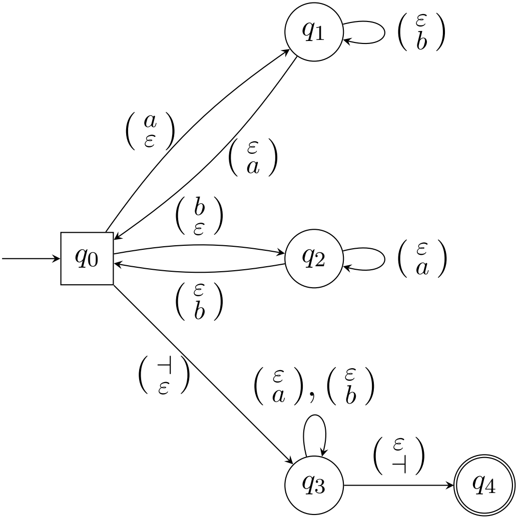
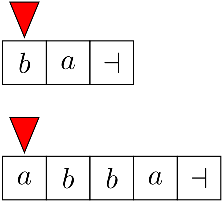




Deterministic Rational Relations

$R = \{(x, y) \mid x$ is a subword of $y\}$





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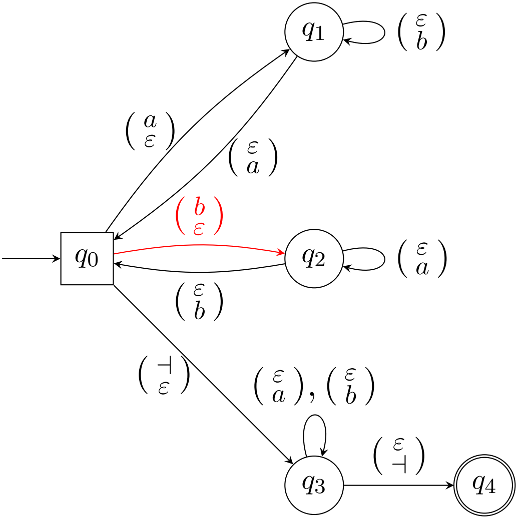

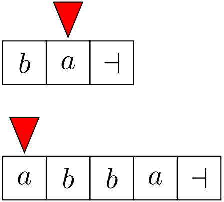

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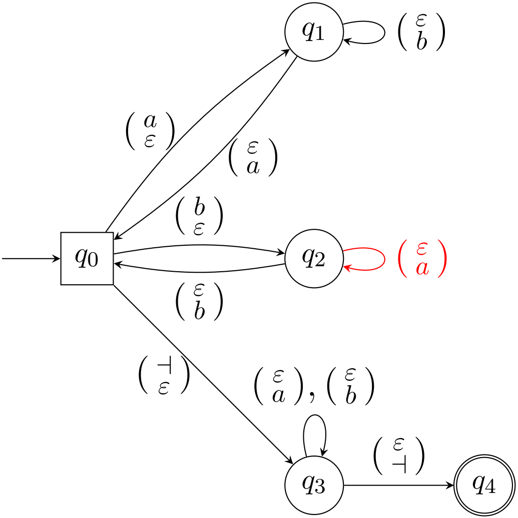

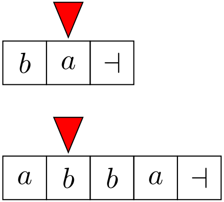

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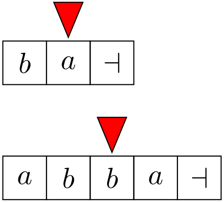

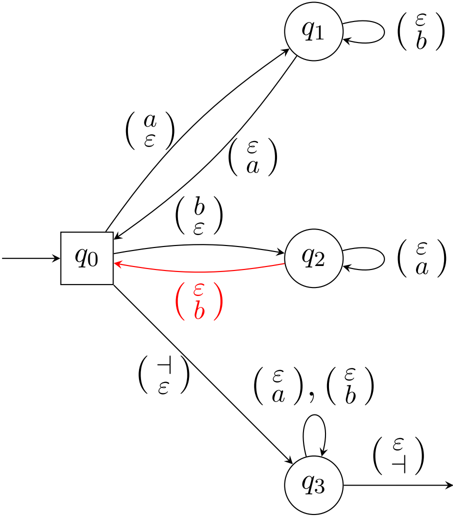

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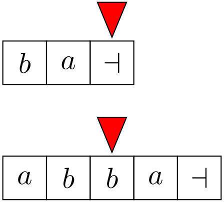

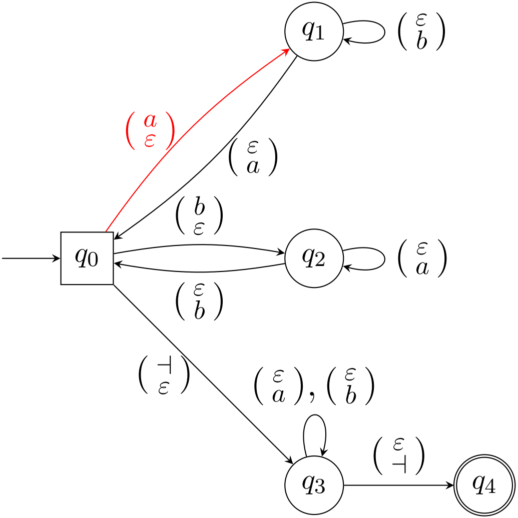

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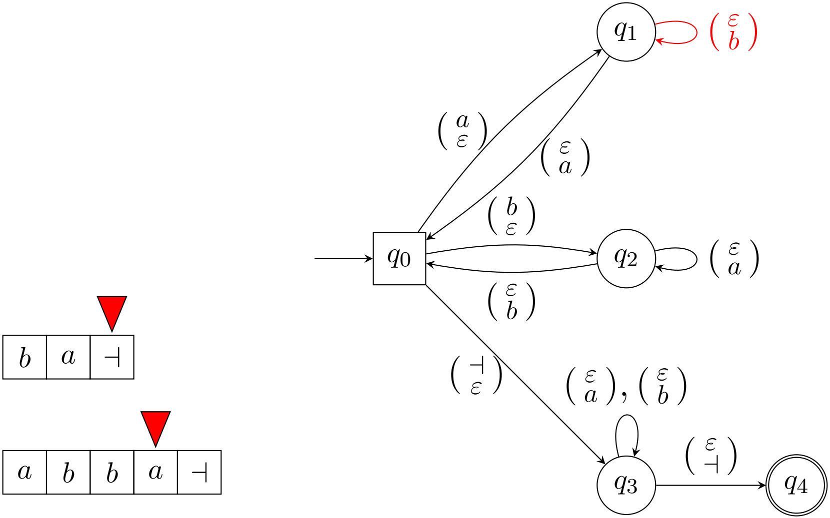

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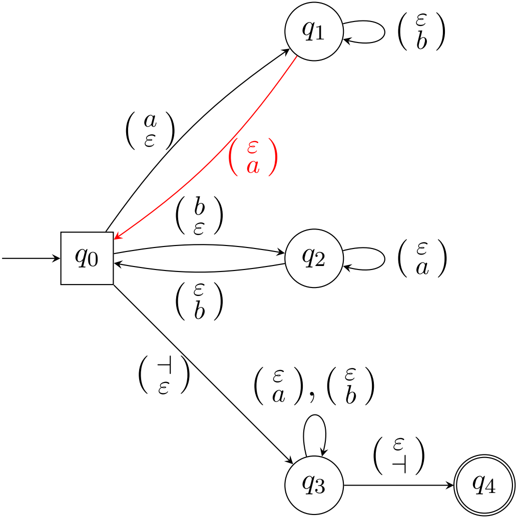

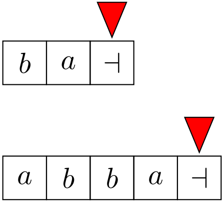

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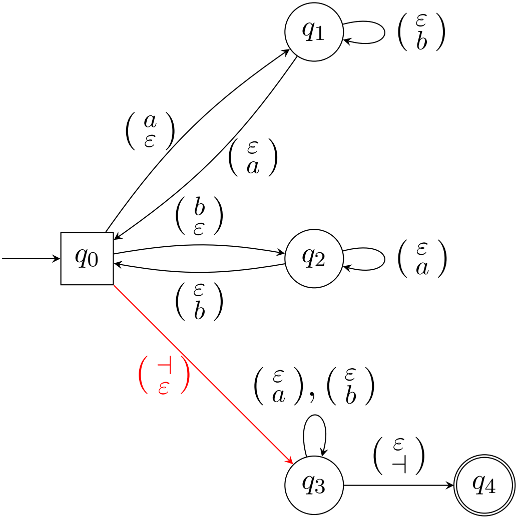

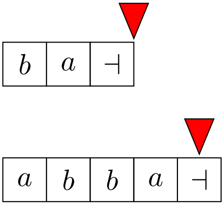

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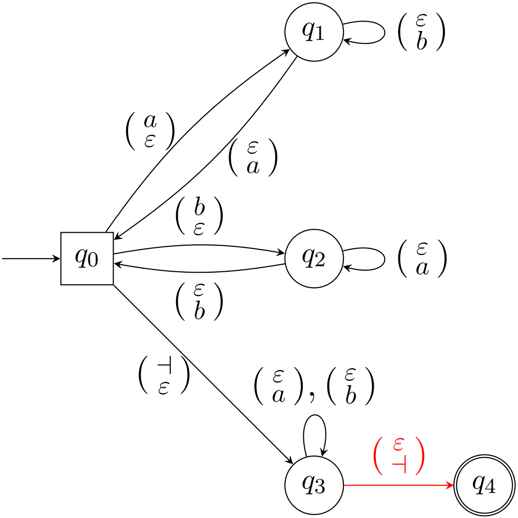

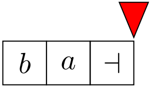

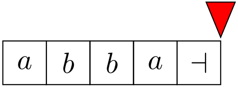

Synchronous Relations

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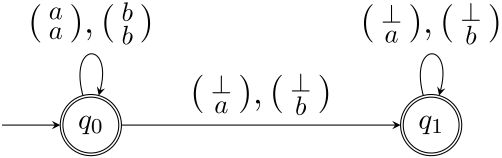

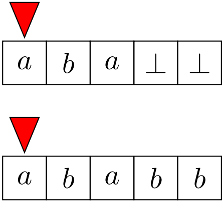

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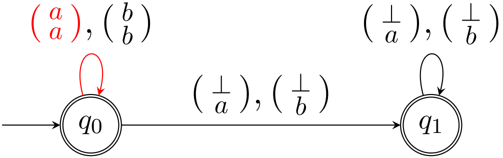

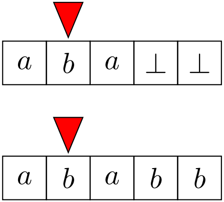

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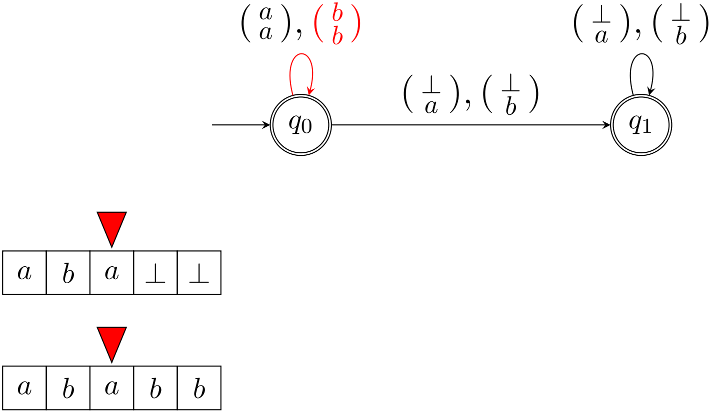

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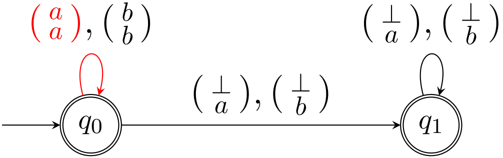

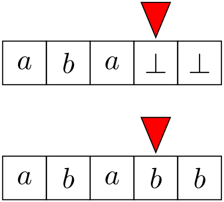

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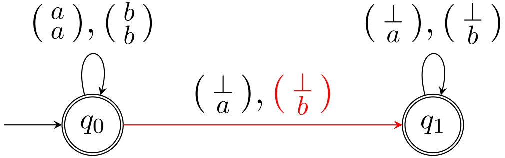

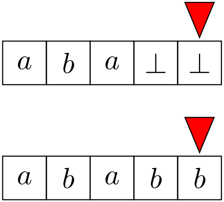

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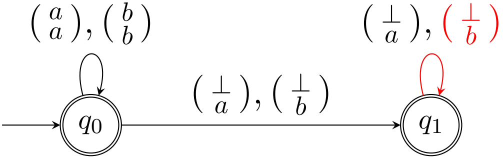
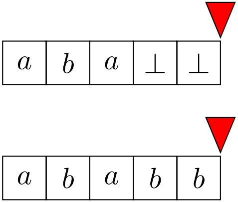




Recognizable Relations

Definition

R is recognizable if it can be written as

$$R = \bigcup_{i=1}^{n} L_{i,1} \times \cdots \times L_{i,k}$$

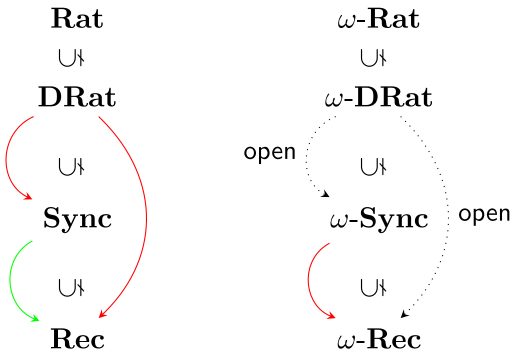
where $L_{i,j}$ are regular languages.

Example

$R = \{(x, y) \mid |x| + |y| \geq 2\}$ can be written as

$$\Sigma^{\geq 2} \times \Sigma^* \quad \cup \quad \Sigma^{\geq 1} \times \Sigma^{\geq 1} \quad \cup \quad \Sigma^* \times \Sigma^{\geq 2}$$

Membership Problems



Recognizability via Finite-Index Equivalences

For k-ary relation R define equivalence relation $x \approx_j^R y$ on words such that for all $(z_1, \ldots, z_{k-1})$:

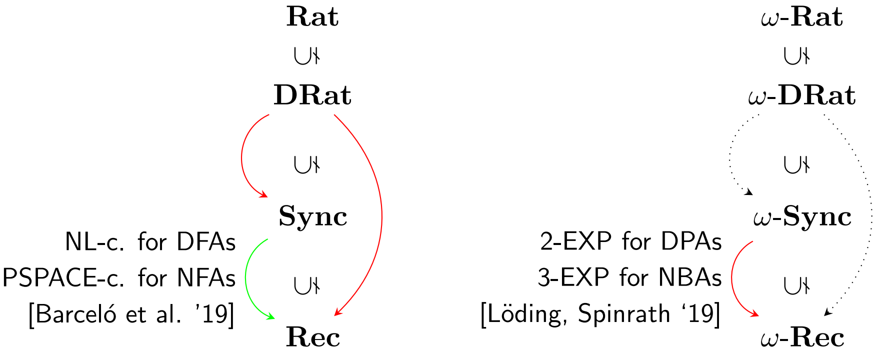
$$(z_1, \ldots, z_{j-1}, x, z_j, \ldots, z_{k-1}) \in R \iff (z_1, \ldots, z_{j-1}, y, z_j, \ldots, z_{k-1}) \in R$$

Proposition

Let $R \in \mathbf{Rat} \cup \omega\text{-}\mathbf{Sync}$. The following are equivalent:

- *R is (ω-)recognizable.*
- *$\approx_j^R$ has finite index for all $j \in [1, k-1]$.*

Deciding ω-Recognizability in ω-**Sync**



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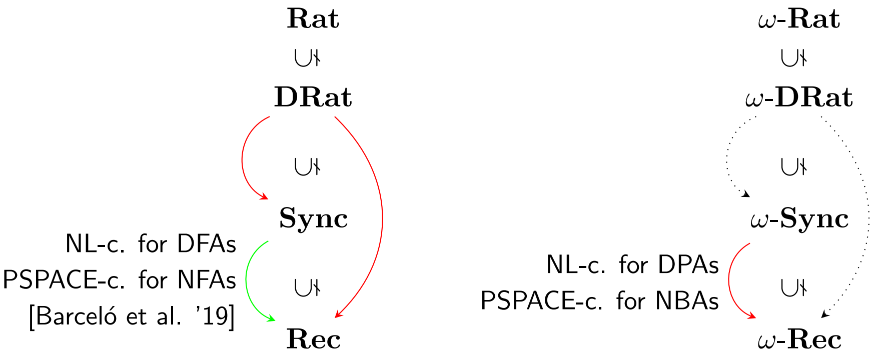


Proof Sketch

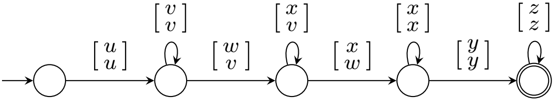
- Given a (non)det. automaton for R, we can compute a nondet. automaton for $\not\approx_j^R$ in logspace (polynomial space).
- R is ω-recognizable iff $\approx_j^R$ has finite index for all $j < k$.
- $\approx_j^R$ has infinite index iff $\not\approx_j^R$ has an infinite clique.
- How to check for an infinite clique in an arbitrary ω-synchronous relation is a longstanding open problem, but:

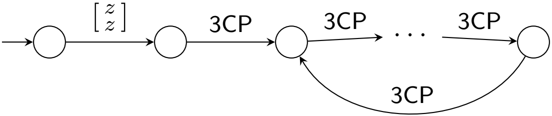
Theorem

It is NL*-complete to decide, given a nondet. Büchi automaton for an ω-synchronous co-equivalence relation $\bar{E}$, whether $\bar{E}$ has an infinite clique.*

Proof Sketch of Theorem

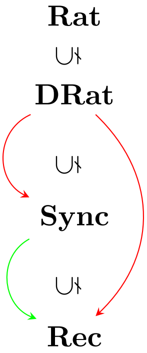
$\bar{E}$ contains an infinite clique iff the automaton for $\bar{E}$ contains one of the following patterns:





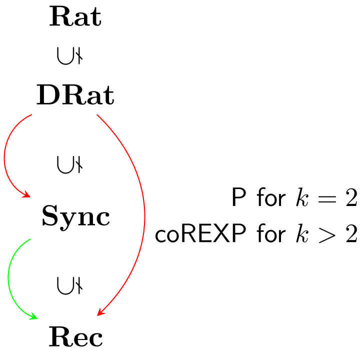
The existence of these patterns can be checked in NL.

Deciding Recognizability in $\mathbf{DRat}$



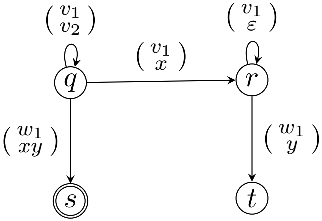
2-EXP for $k = 2$ [Valiant '75]
decidable for $k > 2$ [Carton et al. '06]

Deciding Recognizability in $\mathbf{DRat}$



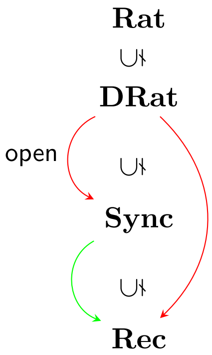
Proof Sketch for $k = 2$

To check if $\approx_1^R$ as infinite index, it suffices to check if the automaton for R contains the pattern:

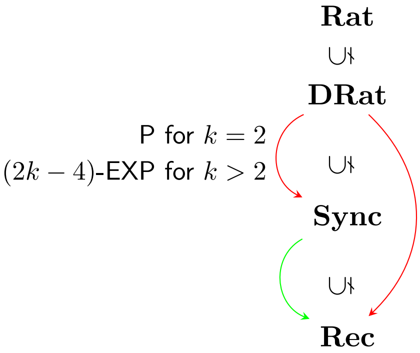


Detection of this pattern can be reduced to the inequivalence problem for binary det. rational relations which is in P.

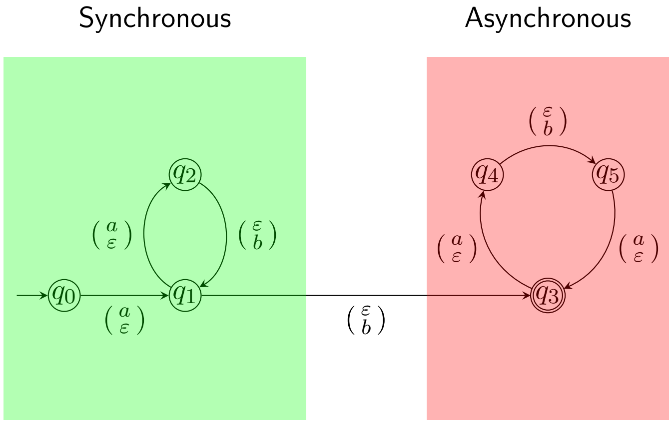
Deciding Synchronicity in $\mathbf{DRat}$



Deciding Synchronicity in $\mathbf{DRat}$



Proof Sketch for $k = 2$



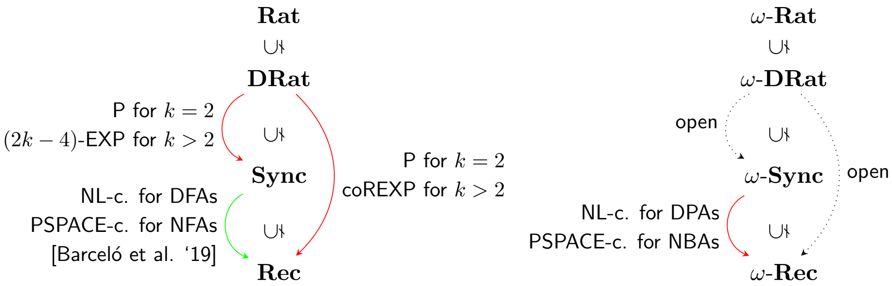
Proof Sketch for $k = 2$

- Bounded delay in synchronous part guarantees synchronicity since symbols that are read ahead can be stored in a queue.
- In the asynchronous part there would be no bound on the queue size.
- We reduce to recognizability of binary relations in $\mathbf{DRat}$:

Lemma

R is synchronous if and only if for each reachable state q in the asynchronous part, the relation accepted from q is recognizable.

Summary



Rat
DRat
Sync
Rec
P for $k = 2$
$(2k-4)$-EXP for $k > 2$
NL-c. for DFAs
PSPACE-c. for NFAs
[Barceló et al. '19]
P for $k = 2$
coREXP for $k > 2$
ω-Rat
ω-DRat
ω-Sync
ω-Rec
open
open
NL-c. for DPAs
PSPACE-c. for NBAs