On the computational expressivity of (circular) proofs with fixed points

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Fixed point logics

- Fixed points are pervasive in mathematics and computer science:
  - **Logic:** model (co)inductive reasoning
  - **Computation:** represent (co)recursion mechanisms

- **Fixed point logics** as (finitary and circular) type systems:
  - μLJ $=$ LJ $+$ least/greatest fixed points [Cla]
  - μMALL $=$ MALL $+$ least/greatest fixed points [Bae12, BDS]
  - μLL $=$ LL$+$ least/greatest fixed points [EJ]

- **Fundamental question:**

  What functions do (circular) proof systems with fixed points represent?

- **Our result:** μLJ and CμLJ represent the same class of functions, namely those provably recursive in the subsystem of second-order arithmetic Π^1_2-CA$_0$.

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1 Systems with fixed points

2 Overview of the results

3 Concluding remarks

Proof system μLJ and its circular version CμLJ

- Formulas:

Least fixed point

Greatest fixed point

$$\sigma, \tau ::= X \mid 1 \mid \sigma \to \tau \mid \sigma \times \tau \mid \sigma + \tau \mid \mu X \sigma \mid \nu X \sigma$$

X occurs positive in σ

- Functions on natural numbers represented by proofs of $\boxed{\mu\text{LJ}}$ via cut elimination:
  - Natural numbers: $N := \mu X(1 + X)$ $\underline{n}$ construed using $n + 1$ units
  - Addition:

$$\text{add}(0, y) = y$$
$$\text{add}(x + 1, y) = \text{add}(x, y) + 1$$

$$\underline{\text{add}} := \text{unfold}\;\dfrac{\text{case}\;\dfrac{\text{unit}\;\dfrac{\text{id}\;\dfrac{}{N \vdash N}}{1, N \vdash N} \qquad \text{succ}\;\dfrac{}{N \vdash N}}{1 + N, N \vdash N}}{N, N \vdash N}$$

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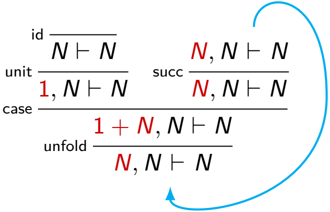

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The theories of arithmetic μPA and Π^1_2-CA$_0$

- μPA := PA + least (and greatest) fixed points (essentially in [Mö02, Tup04])

Set variables   Least fixed point (X positive in ϕ)

$$\phi \quad := \quad t = u \mid t < u \mid t \in X \mid t \in \mu X x \phi \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \forall x \phi \mid \exists x \psi$$

Extension of PA by the following axioms for $\phi(X, x)$, $\psi(x)$:

  - $\mu X x \phi$ **pre-fixed point** : $\forall y(\phi(\mu X x\phi, y) \to y \in \mu X x \phi)$
  - $\mu X x \phi$ **least pre-fixed point** : $\forall x(\phi(\psi, x) \to \psi(x)) \to (y \in \mu X x \phi \to \psi(y))$

- Π^1_2-CA$_0$ = PA$_2$ with comprehension scheme restricted to Π^1_2 formulas:

$$\exists X \forall x(\phi(x) \leftrightarrow x \in X) \qquad X \notin FV(\phi) \qquad \phi \in \Pi^1_2$$

- **Theorem** [Mö02]: Π^1_2-CA$_0$ is arithmetically conservative over μPA.

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1. Systems with fixed points
2. **Overview of the results**
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Our results in a nutshell

- **Theorem:** μLJ and CμLJ represent the same class of functions, namely those functions provably recursive in Π^1_2-CA$_0$.

$$\begin{tikzcd}
\textbf{Theories of arithmetic} & \mu\mathsf{PA} \arrow[d, "\text{realisability argument}"'] & \Pi^1_2\text{-}\mathsf{CA}_0 \arrow[l, "{[\text{Mö02}]}"'] \\
\textbf{Proof systems} & \mu\mathsf{LJ} \arrow[r, red, "\text{simulation}"'] & \mathsf{C}\mu\mathsf{LJ} \arrow[u, "\text{formalisation of totality argument}"']
\end{tikzcd}$$

Translation compiling (co)iterators to cycles

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Via hereditary recursive predicates:

If $\mathcal{D}$ is a circular proof of τ in CμLJ$^-$ *then* $\mathcal{D}^{HR} \in \tau^{HR}$

Novel reverse mathematics of fixed point theorems

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Π^1_2-CA_0 is arithmetically conservative over μPA

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μ and ν viewed under second-order encoding (μHA $\subset$ PA$_2$) [BT21]:
If $\mu\mathrm{HA} \vdash \phi$ then there is proof $\mathcal{D}$ of ϕ^R in μLJ such that $\mathcal{D}^R$ realises ϕ.

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Conclusion and future works

- We have settled the problem of computational expressivity of μLJ and CμLJ.
- The result can be extend to μMALL [Bae12] and its circular version CμMALL by adapting standard translation $(\_)^\bullet : \mathrm{LJ} \to \mathrm{LL}$ with $(\sigma \to \tau)^\bullet = !\sigma^\bullet \multimap \tau^\bullet$ using encoding of exponentials [Bae12]:

$$?\sigma := \mu X(\bot \oplus (X \parr X) \oplus \sigma) \qquad !\sigma := \nu X(1 \,\&\, (X \otimes X) \,\&\, \sigma)$$

- **Future works:**
  - Restricting μLJ and CμLJ to **strictly** positive fixed points (μ and ν never bind variables under the left of implication)
  - Generalisation of $\mathrm{CID}_{<\omega} = \mathrm{ID}_{<\omega}$ [DM23] $\Longrightarrow$ $\boxed{\mathrm{C}\mu\mathrm{PA} \overset{?}{=} \mu\mathrm{PA}}$

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Thank you!
Questions?

References:

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Appendix

4 Proof systems

5 Theories of arithmetic

6 Overview of the results

The finitary proof system μLJ

- Formulas:

$$\sigma, \tau ::= X \mid 1 \mid \sigma \to \tau \mid \sigma \times \tau \mid \sigma + \tau \mid \mu X\sigma \mid \nu X\sigma$$

Condition: X occurs positively in σ for $\mu X\sigma$ and $\nu X\sigma$.

- Examples: $B := 1 + 1$

$$\mu X(X \to B) \qquad \mu X(B \to X) \qquad \mu X((X \to B) \to B)$$

- Inference rules:

$$\mu_r\,\frac{\Gamma \vdash \sigma[\mu X\sigma/X]}{\Gamma \vdash \mu X\sigma} \quad \mu_l\,\frac{\Gamma, \sigma[\tau/X] \vdash \tau}{\Gamma, \mu X\sigma \vdash \tau} \qquad \nu_r\,\frac{\Gamma, \tau \vdash \sigma[\tau/X]}{\Gamma, \tau \vdash \nu X\sigma} \quad \nu_l\,\frac{\Gamma, \sigma[\nu X\sigma/X] \vdash \tau}{\Gamma, \nu X\sigma \vdash \tau}$$

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Some intuition

- Positivity condition on $\mu X \sigma(X)$ and $\nu X \sigma(X)$ $\Longrightarrow$ monotonicity
- Interpreting the inference rules:

$$\sigma(\mu X\sigma) \to \mu X\sigma \qquad (\sigma(\tau) \to \tau) \to \mu X\sigma \to \tau$$

$\mu X\sigma$ is a pre-fixed point $\mu X\sigma$ is the meet of all pre-fixed points

$$\nu X\sigma \to \sigma(\nu X\sigma) \qquad (\tau \to \sigma(\tau)) \to \tau \to \nu X\sigma$$

$\nu X\sigma$ is a post-fixed point $\nu X\sigma$ is the join of all post-fixed points

- **Knaster-Tarski's theorem:**

$$\begin{aligned} \text{least fixed point} &= \text{meet of all pre-fixed points} \\ \text{greatest fixed point} &= \text{join of all post-fixed points} \end{aligned}$$

Encodings indutive data types in μLJ

- Natural numbers: $\underline{n}$ construed using $n+1$ units

$$N := \mu X(1+X) \qquad \underline{0} := \mu_r\,\dfrac{\overleftarrow{+}_r\,\dfrac{1_r\,\dfrac{}{\vdash 1}}{\vdash 1+N}}{\vdash N} \qquad \underline{n+1} := \mu_r\,\dfrac{\overrightarrow{+}_r\,\dfrac{\underset{\vdash N}{\overset{\underline{n}}{\bigtriangledown}}}{\vdash 1+N}}{\vdash N}$$

- Addition:

$$\begin{aligned}\mathrm{add}(0,y) &= y\\ \mathrm{add}(x+1,y) &= \mathrm{add}(x,y)+1\end{aligned} \qquad \mathrm{add} := \mu_l\,\dfrac{+_l\,\dfrac{1_l\,\dfrac{\mathrm{id}\,\dfrac{}{N \vdash N}}{1, N \vdash N} \qquad \mathrm{succ}\,\dfrac{}{N \vdash N}}{1+N, N \vdash N}}{N, N \vdash N}$$

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- Streams of natural numbers:

$$S := \nu X(N \times X) \qquad hd := \nu_l\,\dfrac{\times_l\,\dfrac{\mathrm{id}\,\dfrac{}{N \vdash N}}{N \times S \vdash N}}{S \vdash N} \qquad tl := \nu_l\,\dfrac{\times_l\,\dfrac{\mathrm{id}\,\dfrac{}{S \vdash S}}{N \times S \vdash S}}{S \vdash N}$$

- Unfolding: $n, \alpha, f \mapsto f(0) :: f(1) :: \ldots :: f(n-1) :: \alpha(n) :: \alpha(n+1) :: \ldots$

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The circular proof system CμLJ

- Non-wellfounded proofs generated by the inference rules:

$$\mu_r\,\frac{\Gamma \vdash \sigma[\mu X\sigma/X]}{\Gamma \vdash \mu X\sigma} \quad \mu_l\,\frac{\Gamma, \sigma[\mu X\sigma/X] \vdash \tau}{\Gamma, \mu X\sigma \vdash \tau} \qquad \nu_r\,\frac{\Gamma \vdash \sigma[\nu X\sigma/X]}{\Gamma \vdash \nu X\sigma} \quad \nu_l\,\frac{\Gamma, \sigma[\nu X\sigma/X] \vdash \tau}{\Gamma, \nu X\sigma \vdash \tau}$$

- **Progressiveness** condition:
  - thread = maximal FL-subformula path along a branch
  - progressing proof = any infinite branch has a thread that is infinitely often principal and the smallest infinitely often principal formula is μ-formula on the LHS or a ν-formula on the RHS (of $\vdash$)
  - progressiveness allows to **maintain logical consistency** ($\nu \neq \mu$)

- **Regularity** condition:
  - circular proof = finitely many distinct subproofs
  - circular proofs admit **finite presentation** (finite trees with backpointers)

- **Circular system** CμLJ = circular and progressing non-wellfounded proofs.

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  - progressiveness allows to **maintain logical consistency** ($\nu \neq \mu$)

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  - circular proof = finitely many distinct subproofs
  - circular proofs admit **finite presentation** (finite trees with backpointers)

- **Circular system** CμLJ = circular and progressing non-wellfounded proofs.

The circular proof system CμLJ

- Non-wellfounded proofs generated by the inference rules:

$$\mu_r\,\frac{\Gamma \vdash \sigma[\mu X\sigma/X]}{\Gamma \vdash \mu X\sigma} \quad \mu_l\,\frac{\Gamma, \sigma[\mu X\sigma/X] \vdash \tau}{\Gamma, \mu X\sigma \vdash \tau} \qquad \nu_r\,\frac{\Gamma \vdash \sigma[\nu X\sigma/X]}{\Gamma \vdash \nu X\sigma} \quad \nu_l\,\frac{\Gamma, \sigma[\nu X\sigma/X] \vdash \tau}{\Gamma, \nu X\sigma \vdash \tau}$$

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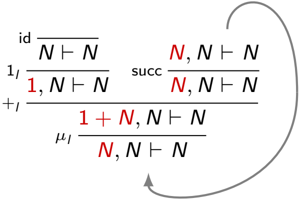
Encodings indutive data types in CμLJ

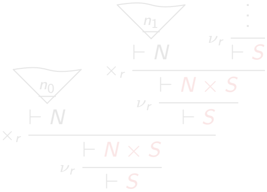
- Addition:

$$\mathsf{add}(0, y) = y$$
$$\mathsf{add}(x + 1, y) = \mathsf{add}(x, y) + 1$$



$$\mathsf{add} := \mu_l\,\dfrac{+_l\,\dfrac{1_l\,\dfrac{\text{id}\,\dfrac{}{N \vdash N}}{1, N \vdash N} \qquad \text{succ}\,\dfrac{N, N \vdash N}{N, N \vdash N}}{1 + N, N \vdash N}}{N, N \vdash N}$$

- Streams of natural numbers:



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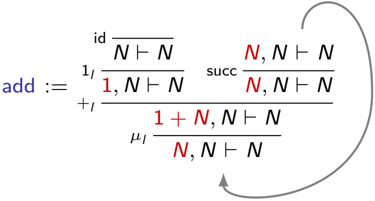

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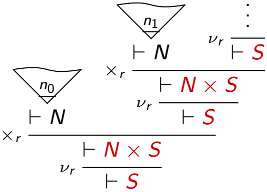

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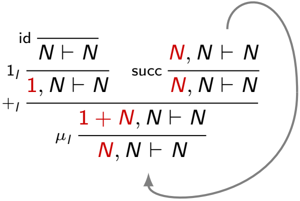

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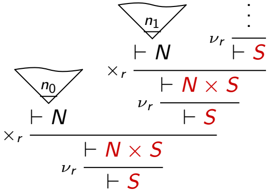

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4 Proof systems

5 Theories of arithmetic

6 Overview of the results

The theory μPA

- Formulas: closure of $\mathcal{L}_{\mathsf{PA}}$ under fixed point operators

$$\phi \quad := \quad t = u \mid t < u \mid t \in X \mid t \in \mu X \lambda x \phi \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \forall x \phi \mid \exists x \psi$$

Condition: X occurs positively in ϕ for $t \in \mu X \lambda x \phi$.

- Semantically:
  - if $\phi(X, x)$ then $\lambda x \phi$ interpreted as $\{n \in \mathbb{N} \mid \mathcal{N} \models \phi(\underline{n})\} \in \wp(\mathbb{N})$
  - This induces a **monotone** function:

$$\begin{aligned} \wp(\mathbb{N}) &\to \wp(\mathbb{N}) \\ A &\mapsto \{n \in \mathbb{N} \mid \mathcal{N} \models \phi(A, \underline{n})\} \end{aligned}$$

  - $\mu X.\lambda x.\phi$ interpreted as the least fixed point of the above function.

- Theory μPA = extension of PA by the following axioms for $\phi(X, x)$, $\psi(x)$:
  - $\forall y(\phi(\mu X \lambda x \phi, y) \to y \in \mu X \lambda x \phi)$
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The theory Π^1_2-CA_0 and conservativity properties

- Π^1_2-CA_0 = PA_2 with comprehension scheme restricted to Π^1_2 formulas:

$$\exists X \forall x(\phi(x) \leftrightarrow x \in X) \qquad X \notin FV(\phi) \qquad \phi \in \Pi^1_2$$

- **Theorem** [Mö02]: Π^1_2-CA_0 is arithmetically conservative over μPA.

- Theorem [Tup04]: μPA is Π^0_2-conservative over μHA.

- **Consequence:** μPA, μHA and Π^1_2-CA_0 prove the totality of the same functions on natural numbers.

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- **Theorem** [Mö02]: Π^1_2-CA_0 is arithmetically conservative over $\mu\mathsf{PA}$.

- Theorem [Tup04]: $\mu\mathsf{PA}$ is Π^0_2-conservative over $\mu\mathsf{HA}$.

- **Consequence:** $\mu\mathsf{PA}$, $\mu\mathsf{HA}$ and Π^1_2-CA_0 prove the totality of the same functions on natural numbers.

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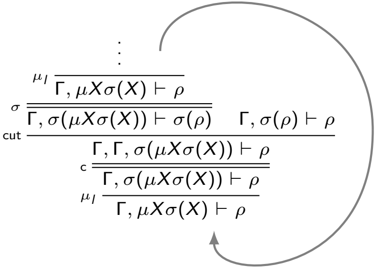
Back to our grand tour diagram

$$\begin{tikzcd}
\textbf{Theories of arithmetic:} & \mu\mathsf{HA} \arrow[d, "\text{realisability}"] & \mu\mathsf{PA} \arrow[l, "\text{[Tup04]}"'] & \Pi^1_2\text{-}\mathsf{CA}_0 \arrow[l, "\text{[Mö02]}"'] \\
\textbf{Proof systems} & \mu\mathsf{LJ} \arrow[r, "\text{simulation}"'] & \mathsf{C}\mu\mathsf{LJ} \arrow[r, "\text{negative translation}"'] & \mathsf{C}\mu\mathsf{LJ}^- \arrow[u, "\text{totality argument}"]
\end{tikzcd}$$

Simulation of μLJ into CμLJ

- **Idea:** compiling (co)iteration to loops



$$\mu_l\,\frac{\Gamma, \sigma(\rho) \vdash \rho}{\Gamma, \mu X\sigma(X) \vdash \rho} \qquad \mapsto \qquad \mu_l\,\frac{\mathsf{c}\,\dfrac{\mathsf{cut}\,\dfrac{\sigma\,\dfrac{\mu_l\,\dfrac{\vdots}{\Gamma, \mu X\sigma(X) \vdash \rho}}{\Gamma, \sigma(\mu X\sigma(X)) \vdash \sigma(\rho)} \qquad \Gamma, \sigma(\rho) \vdash \rho}{\Gamma, \Gamma, \sigma(\mu X\sigma(X)) \vdash \rho}}{\Gamma, \sigma(\mu X\sigma(X)) \vdash \rho}}{\Gamma, \mu X\sigma(X) \vdash \rho}$$

- Dually for ν_r.

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Translation from CμLJ to CμLJ$^-$

- CμLJ$^-$ $= \{N, \times, \to, \mu\}$-fragment of C$\mu$LJ
- Kolmogorov-style negative translation with $\neg\sigma := \sigma \to N$ (Friedman-Dragalin):

$$
\begin{aligned}
X^N &:= \neg\neg X \\
1^N &:= \neg N \\
(\sigma \times \tau)^N &:= \neg\neg(\sigma^N \times \tau^N) \\
(\sigma \to \tau)^N &:= \neg\neg(\sigma^N \to \tau^N) \\
(\sigma + \tau)^N &:= \neg(\neg\sigma^N \times \neg\tau^N) \\
(\nu X \sigma)^N &:= \neg\neg\neg\mu X \neg\sigma^N[\neg X / X] \\
(\mu X \sigma)^N &:= \neg\neg\mu X \sigma^N
\end{aligned}
$$

- **NB:** Negative translation simplifies our results (but we need **non-strictly positive** fixed points). Gödel-Gentzen-style translation **does not preserve progressiveness**.

- **Theorem:** CμLJ and CμLJ$^-$ represent the same functions on natural numbers.

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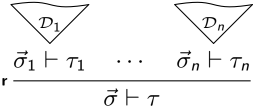
Rules as combinators, proofs as (co)terms

- Rules into combinators:

$$\text{r}\,\frac{\vec{\sigma}_1 \vdash \tau_1 \qquad \vec{\sigma}_n \vdash \tau_n}{\vec{\sigma} \vdash \tau} \qquad \rightsquigarrow \qquad \mathbf{r} : (\vec{\sigma}_1 \to \tau_1) \to \ldots \to (\vec{\sigma}_n \to \tau_n) \to \vec{\sigma} \to \tau$$

- Circular proofs into (untyped) coterms:

$$\mathcal{D} \in \mathsf{C}\mu\mathsf{LJ}^- \qquad \mapsto \qquad t_{\mathcal{D}} \in \mathsf{coTer}$$



$$\text{r}\,\frac{\overset{\mathcal{D}_1}{\vec{\sigma}_1 \vdash \tau_1} \quad \cdots \quad \overset{\mathcal{D}_n}{\vec{\sigma}_n \vdash \tau_n}}{\vec{\sigma} \vdash \tau} \qquad \rightsquigarrow \qquad \mathbf{r}\; t_{\mathcal{D}_1}\; \ldots\; t_{\mathcal{D}_n}$$

- Equational theory $=_{\mathsf{cut}}$ on coterms essentially induced by cut elimination.

A totality argument for proofs of CμLJ$^-$

- Candidate A is subset of coTer closed under $=_{\mathsf{cut}}$.

- Type structure: with each formula σ we associate a set $|\sigma| \subseteq$ coTer (the *domain of* σ):

$$
\begin{aligned}
|A| &:= A & & A \text{ candidate} \\
|N| &:= \{t \mid \exists n \in \mathbb{N}.\, t =_{\mathsf{cut}} \underline{n}\} & & \\
|\sigma \to \tau| &:= \{t \mid \forall s \in |\sigma|.\, ts \in |\tau|\} & & \\
|\mu X \sigma(X)| &:= \bigcap_{|\sigma(A)| \subseteq A} A \text{ candidate} & & |\sigma(\cdot)| := A \mapsto |\sigma(A)|
\end{aligned}
$$

  μ-formulas as least fixed points: $|\mu X \sigma(X)| = \bigcup_{\alpha \in \mathrm{Ord}} |\sigma^{\alpha}(\varnothing)|$. Important for logical complexity.

- **Theorem**: If $\mathcal{D}$ is a circular proof of τ in CμLJ$^-$ then $t_{\mathcal{D}} \in |\tau|$.

- **Consequence**: CμLJ$^-$ represents only total functions. We can formalise metamathematically the totality argument for CμLJ$^-$ in Π^1_2-CA$_0$.

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- **Theorem**: If $\overset{\mathcal{D}}{\vec{\sigma} \vdash \tau}$ in CμLJ$^-$ then, for every $\vec{s} \in |\vec{\sigma}|$, we have $t_{\mathcal{D}}\, \vec{s} \in |\tau|$.

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  - By **progressiveness** B contains a thread (infinite FL-subformula path) where some $\mu X \sigma(X)$ (in negative position) **unfolds infinitely often**.
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  - By **progressiveness** B contains a thread (infinite FL-subformula path) where some $\mu X \sigma(X)$ (in negative position) **unfolds infinitely often**.
  - **Assign ordinals** approximating the "critical" $\mu X \sigma(X)$, obtaining a non-increasing ordinal assignment along B. In particular, at any unfolding of the "critical" $\mu X \sigma(X)$ the corresponding ordinal strictly decreases. **Contradiction**.

Idea of the totality argument

- **Theorem**: If $\overset{\mathcal{D}}{\vec{\sigma} \vdash \tau}$ in CμLJ^{-} then, for every $\vec{s} \in |\vec{\sigma}|$, we have $t_{\mathcal{D}}\, \vec{s} \in |\tau|$.

- **Idea of the proof.** By contradiction:
  - Assume $t_{\mathcal{D}}\, s_1 \ldots s_n \notin |\tau|$ for some $s_1 \in |\sigma_1|, \ldots, s_n \in |\sigma_n|$. If

    $$\mathcal{D} \quad = \quad {\scriptstyle\mathsf{r}}\, \dfrac{\overset{\mathcal{D}_1}{\vec{\sigma}_1 \vdash \tau_1} \qquad \cdots \qquad \overset{\mathcal{D}_n}{\vec{\sigma}_n \vdash \tau_n}}{\vec{\sigma} \vdash \tau}$$

    with $\vec{s} \in |\vec{\sigma}|$ s.t. $t_{\mathcal{D}}\, \vec{s} \notin |\tau|$, then there is a proof $\mathcal{D}_i$ of $\vec{\sigma}_i \vdash \tau_i$ some inputs $\vec{s}_i \in |\vec{\sigma}_i|$ such that $t_{\mathcal{D}_i}\, \vec{s}_i \notin |\tau_i|$. Construct **infinite "non-total" branch** B in $\mathcal{D}$.
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\textbf{Theories of arithmetic:} & \mu\mathsf{HA} \arrow[d, "\text{realisability}"] & \mu\mathsf{PA} \arrow[l, "{[\text{Tup04}]}"'] & \Pi^1_2\text{-}\mathsf{CA}_0 \arrow[l, "{[\text{Mö02}]}"'] \\
\textbf{Proof systems} & \mu\mathsf{LJ} \arrow[r, "\text{simulation}"'] & \mathsf{C}\mu\mathsf{LJ} \arrow[r, "\text{negative translation}"'] & \mathsf{C}\mu\mathsf{LJ}^- \arrow[u, "\text{totality argument}"]
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From μHA to μLJ via realisability

- Proofs as (untyped, inductive) terms: $\mathcal{D} \in \mu\mathsf{LJ} \quad \mapsto \quad t_{\mathcal{D}} \in \mathsf{Ter}$

- For any $t \in \mathsf{Ter}$ and $\phi(X_1, \ldots, X_n)$ formula we define:

$$t \; \mathbf{r} \, \phi \quad \text{w.r.t. } \textbf{realisability candidates } R_{X_1}, \ldots, R_{X_n} \subseteq \mathsf{Ter} \times \mathbb{N}$$

$t \; \mathbf{r} \, (\underline{n} \in X)$ if $t \; R_X \; n$

$t \; \mathbf{r} \, (\underline{n} \in \mu X \lambda x \phi)$ if, for all R_X, $t \; \mathbf{r} \; \forall x(\phi(X, x) \to x \in X) \to \underline{n} \in X$

Idea. μHA as a subsystem of HA$_2$:

$$t \in \mu X \lambda x \phi := \forall X(\forall x(\phi(X, x) \to x \in X) \to t \in X)$$

- **Theorem:** If $\mu\mathsf{HA} \vdash \phi$ then there is $\underset{\vdash \mathtt{t}(\phi)}{\mathcal{D}}$ in μLJ such that $t_{\mathcal{D}} \; \mathbf{r} \, \phi$.

- **Consequence:** The provably recursive functions of μHA are representable in μLJ

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Extending our results to other fixed point logics

- The results extend to μMALL [Bae12] and its circular version CμMALL:

$$\begin{tikzcd}
\mu\mathsf{LJ} \arrow[d, "\text{translation}", "(*)"'] & \mathsf{C}\mu\mathsf{LJ} \arrow[l, "\text{this talk}"'] \\
\mu\mathsf{MALL} \arrow[r, "\text{simulation}"'] & \mathsf{C}\mu\mathsf{MALL} \arrow[u, "\text{negative translation}"']
\end{tikzcd}$$

(*) Adapt standard translation $(\_)^\bullet : \mathsf{LJ} \to \mathsf{LL}$ with $(\sigma \to \tau)^\bullet = !\sigma^\bullet \multimap \tau^\bullet$ using encoding of exponentials [Bae12]:

$$?\sigma := \mu X(\bot \oplus (X \parr X) \oplus \sigma) \qquad !\sigma := \nu X(1 \mathbin{\&} (X \otimes X) \mathbin{\&} \sigma)$$

- Similarly for μLL and CμLL (fully fledged linear logic).