On the computational expressivity of (circular) proofs with fixed points

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Fixed point logics

- Fixed points are pervasive in mathematics and computer science:
  - **Logic**: model (co)inductive reasoning
  - **Computation**: represent (co)recursion mechanisms

- Fixed point logics as (finitary and circular) type systems:
  - $\mu\text{LJ} = \text{LJ} + \text{least/greatest fixed points}$ [Cla]
  - $\mu\text{MALL} = \text{MALL} + \text{least/greatest fixed points}$ [Bae12, BDS]
  - $\mu\text{LL} = \text{LL} + \text{least/greatest fixed points}$ [EJ]

- **Fundamental question:**
  
  What functions do (circular) proof systems with fixed points represent?

- **Our result:** $\mu\text{LJ}$ and $C\mu\text{LJ}$ represent the same class of functions, namely those provably recursive in the subsystem of second-order arithmetic $\Pi^1_2\text{-CA}_0$. 
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- **Fixed point logics** as (finitary and circular) type systems:
  - $\mu LJ = LJ + \text{least/greatest fixed points} \ [\text{Cla}]$
  - $\mu MALL = MALL + \text{least/greatest fixed points} \ [\text{Bae12, BDS}]$
  - $\mu LL = LL + \text{least/greatest fixed points} \ [\text{EJ}]$

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1 Systems with fixed points

2 Overview of the results

3 Concluding remarks
Proof system $\mu$LJ and its circular version $C\mu$LJ

Formulas:

\[ \sigma, \tau ::= X \mid 1 \mid \sigma \to \tau \mid \sigma \times \tau \mid \sigma + \tau \mid \mu X \sigma \mid \nu X \sigma \]

Functions on natural numbers represented by proofs of $\mu$LJ via cut elimination:

- Natural numbers: $N := \mu X (1 + X)$
- Addition:
  \[
  \begin{align*}
  \text{id} & : N \vdash N \\
  \text{unit} & : 1, N \vdash N \\
  \text{succ} & : N \vdash N \\
  \text{case} & : 1 + N, N \vdash N \\
  \text{unfold} & : N, N \vdash N
  \end{align*}
  \]
  \[
  \begin{align*}
  \text{add} := & \begin{array}{l}
  \text{id} \quad \text{unit} \quad \text{succ} \\
  \text{add}(0, y) = y \\
  \text{add}(x + 1, y) = \text{add}(x, y) + 1
  \end{array}
  \end{align*}
  \]

- $X$ occurs positive in $\sigma$

- Least fixed point

- Greatest fixed point
Proof system $\mu$LJ and its circular version $C\mu$LJ

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- **Functions on natural numbers** represented by proofs of $\mu$LJ via cut elimination:

  - **Natural numbers:**  
    \[ N ::= \mu X (1 + X) \]
    
    $n$ construed using $n + 1$ units

  - **Addition:**
    \[ \begin{align*}
    \text{add}(0, y) &= y \\
    \text{add}(x + 1, y) &= \text{add}(x, y) + 1
    \end{align*} \]

\[ \text{id} \]

\[ \begin{array}{c}
\text{unit} \\
\text{succ} \\
\text{case} \\
\text{unfold}
\end{array} \]

\[ \begin{align*}
&N \vdash N \\
&1, N \vdash N \\
&N \vdash N \\
&1 + N, N \vdash N \\
&N, N \vdash N
\end{align*} \]
Proof system $\mu$LJ and its circular version $C\mu$LJ

- **Formulas:**

\[
\sigma, \tau ::= X \mid 1 \mid \sigma \rightarrow \tau \mid \sigma \times \tau \mid \sigma + \tau \mid \mu X \sigma \mid \nu X \sigma
\]

[Least fixed point] [Greatest fixed point] [X occurs positive in $\sigma$]

- **Functions on natural numbers** represented by proofs of $C\mu$LJ via cut elimination:

  ▶ **Natural numbers:** $N := \mu X(1 + X)$

  $n$ construed using $n + 1$ units

  ▶ **Addition:**

  \[
  \begin{align*}
  \text{id} & \quad N \vdash N \\
  \text{unit} & \quad 1, N \vdash N \\
  \text{succ} & \quad N, N \vdash N \\
  \text{case} & \quad 1 + N, N \vdash N \\
  \text{unfold} & \quad N, N \vdash N \\
  \end{align*}
  \]

  $\text{add} :=$

  $\text{add}(0, y) = y$

  $\text{add}(x + 1, y) = \text{add}(x, y) + 1$
The theories of arithmetic $\mu PA$ and $\Pi_{2}^{1}$-CA$_{0}$

- $\mu PA := PA +$ least (and greatest) fixed points (essentially in [Mö02, Tup04])

  > Set variables  
  > Least fixed point ($X$ positive in $\phi$)

  \[ \phi := t = u \mid t < u \mid t \in X \mid t \in \mu Xx\phi \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \forall x\phi \mid \exists x\psi \]

  Extension of PA by the following axioms for $\phi(X, x), \psi(x)$:

  - $\mu Xx\phi$ pre-fixed point: $\forall y(\phi(\mu Xx\phi, y) \rightarrow y \in \mu Xx\phi)$
  - $\mu Xx\phi$ least pre-fixed point: $\forall x(\phi(\psi, x) \rightarrow \psi(x)) \rightarrow (y \in \mu Xx\phi \rightarrow \psi(y))$

- $\Pi_{2}^{1}$-CA$_{0} = PA_{2}$ with comprehension scheme restricted to $\Pi_{2}^{1}$ formulas:

  \[ \exists X \forall x(\phi(x) \leftrightarrow x \in X) \quad X \notin \text{FV}(\phi) \quad \phi \in \Pi_{2}^{1} \]

- **Theorem [Mö02]:** $\Pi_{2}^{1}$-CA$_{0}$ is arithmetically conservative over $\mu PA$. 

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The theories of arithmetic $\mu$PA and $\Pi^1_2$-CA$_0$

- $\mu$PA := PA + least (and greatest) fixed points (essentially in [Mö02, Tup04])

Set variables  Least fixed point ($X$ positive in $\phi$)

$\phi := t = u \mid t < u \mid t \in X \mid t \in \mu X x \phi \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \forall x \phi \mid \exists x \psi$

Extension of PA by the following axioms for $\phi(X, x), \psi(x)$:

- $\mu X x \phi$ pre-fixed point: $\forall y(\phi(\mu X x \phi, y) \rightarrow y \in \mu X x \phi)$
- $\mu X x \phi$ least pre-fixed point: $\forall x(\phi(\psi, x) \rightarrow \psi(x)) \rightarrow (y \in \mu X x \phi \rightarrow \psi(y))$

- $\Pi^1_2$-CA$_0$ = PA$_2$ with comprehension scheme restricted to $\Pi^1_2$ formulas:

$$\exists X \forall x(\phi(x) \leftrightarrow x \in X) \quad X \notin \text{FV}(\phi) \quad \phi \in \Pi^1_2$$

- **Theorem [Mö02]**: $\Pi^1_2$-CA$_0$ is arithmetically conservative over $\mu$PA.
The theories of arithmetic $\mu PA$ and $\Pi^1_2$-CA$_0$

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Set variables  Least fixed point ($X$ positive in $\phi$)

$$\phi := t = u \mid t < u \mid t \in X \mid t \in \mu X x \phi \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \forall x \phi \mid \exists x \psi$$

Extension of PA by the following axioms for $\phi(X, x), \psi(x)$:

- $\mu X x \phi$ pre-fixed point: $\forall y (\phi(\mu X x \phi, y) \rightarrow y \in \mu X x \phi)$
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- $\Pi^1_2$-CA$_0 = PA_2$ with comprehension scheme restricted to $\Pi^1_2$ formulas:

$$\exists X \forall x (\phi(x) \leftrightarrow x \in X) \quad X \notin FV(\phi) \quad \phi \in \Pi^1_2$$

- **Theorem [Mö02]**: $\Pi^1_2$-CA$_0$ is arithmetically conservative over $\mu PA$. 

1. Systems with fixed points

2. Overview of the results

3. Concluding remarks
Our results in a nutshell

- **Theorem:** $\mu LJ$ and $C\mu LJ$ represent the same class of functions, namely those functions provably recursive in $\Pi^1_2$-CA$_0$.

---

**Proof systems**

- $\mu PA \leftrightarrow \Pi^1_2$-CA$_0$
- $\mu LJ \leftrightarrow C\mu LJ$
- Simulation
- Formalisation of totality argument
- Realisability argument

**Theories of arithmetic**

- $\mu PA \leftrightarrow [Mö02]$
Our results in a nutshell

- **Theorem:** $\mu$LJ and $C\mu$LJ represent the same class of functions, namely those functions provably recursive in $\Pi^1_2$-CA$_0$.

$\mu$LJ and $C\mu$LJ represent the same class of functions, namely those functions provably recursive in $\Pi^1_2$-CA$_0$.

Via hereditary recursive predicates:

*If $D$ is a circular proof of $\tau$ in $C\mu$LJ$^-$ then $D^{HR} \in \tau^{HR}$*

Novel reverse mathematics of fixed point theorems
Our results in a nutshell

**Theorem:** $\mu LJ$ and $C\mu LJ$ represent the same class of functions, namely those functions provably recursive in $\Pi^1_2$-$CA_0$. 

\[ \Pi^1_2$-$CA_0 \text{ is arithmetically conservative over } \mu PA \]

---

**Theories of arithmetic**

- $\mu PA$
- $\Pi^1_2$-$CA_0$

**Proof systems**

- $\mu LJ$
- $C\mu LJ$

---

Simulation

Formalisation of totality argument

Realisability argument

[\text{Mö02}]
Our results in a nutshell

**Theorem:** \( \mu \text{LJ} \) and \( C \mu \text{LJ} \) represent the same class of functions, namely those functions provably recursive in \( \Pi^1_2\text{-CA}_0 \).

\[ \mu \text{ and } \nu \text{ viewed under second-order encoding (} \mu \text{HA } \subset \text{PA}_2) \text{ [BT21]:} \]

*If \( \mu \text{HA} \vdash \phi \) then there is proof \( D \) of \( \phi^R \) in \( \mu \text{LJ} \) such that \( D^R \) realises \( \phi \).*

---

**Proof systems**

Theories of arithmetic

\( \mu \text{PA} \leftarrow [\text{Mö02}] \rightarrow \Pi^1_2\text{-CA}_0 \)

realisability argument

formalisation of totality argument

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\( \mu \text{PA} \leftarrow [\text{Mö02}] \rightarrow \Pi^1_2\text{-CA}_0 \)

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**Proof systems**

\( \mu \text{LJ} \rightarrow \text{simulation} \rightarrow C \mu \text{LJ} \)
1 Systems with fixed points

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3 Concluding remarks
Conclusion and future works

- We have settled the problem of computational expressivity of $\mu$LJ and $C_\mu$LJ.

- The result can be extend to $\mu$MALL [Bae12] and its circular version $C_\mu$MALL by adapting standard translation $(\_)_\bullet : LJ \rightarrow LL$ with $(\sigma \rightarrow \tau)_\bullet = !\sigma_\bullet \rightarrow \tau_\bullet$ using encoding of exponentials [Bae12]:

  $$\ ?\sigma := \mu X (\bot \oplus (X \not\otimes X) \oplus \sigma) \quad !\sigma := \nu X (1 \& (X \otimes X) \& \sigma)$$

Future works:

- Restricting $\mu$LJ and $C_\mu$LJ to strictly positive fixed points ($\mu$ and $\nu$ never bind variables under the left of implication)

- Generalisation of $CID_{<\omega} = ID_{<\omega}$ [DM23] $\Rightarrow C_\mu PA \not\models \mu PA$
Conclusion and future works

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$$\ ?\sigma := \mu X(\bot \oplus (X \leftrightarrow X) \oplus \sigma) \quad !\sigma := \nu X(1 & (X \otimes X) & \sigma)$$

Future works:

- Restricting $\mu$LJ and $C_\mu$LJ to strictly positive fixed points ($\mu$ and $\nu$ never bind variables under the left of implication)

- Generalisation of $\text{CID}_{<\omega} = \text{ID}_{<\omega}$ [DM23] $\implies$ $C_\mu\text{PA} \equiv ?\mu\text{PA}$
Thank you!
Questions?

References:

Appendix
4 Proof systems

5 Theories of arithmetic

6 Overview of the results
The finitary proof system $\mu LJ$

- **Formulas:**

  \[
  \sigma, \tau ::= X \mid 1 \mid \sigma \rightarrow \tau \mid \sigma \times \tau \mid \sigma + \tau \mid \mu X \sigma \mid \nu X \sigma
  \]

  **Condition:** $X$ occurs positively in $\sigma$ for $\mu X \sigma$ and $\nu X \sigma$.

- **Examples:** $B ::= 1 + 1$

  \[
  \mu X(X \rightarrow B) \quad \mu X(B \rightarrow X) \quad \mu X((X \rightarrow B) \rightarrow B)
  \]

- **Inference rules:**

  \[
  \frac{\Gamma \vdash \sigma[\mu X \sigma/X]}{\mu r} \quad \frac{\Gamma, \sigma[\tau/X] \vdash \tau}{\mu l} \quad \frac{\Gamma, \tau \vdash \sigma[\tau/X]}{\nu r} \quad \frac{\Gamma, \tau \vdash \nu X \sigma}{\nu l}
  \]

  \[
  \frac{\Gamma \vdash \mu X \sigma}{\mu r} \quad \frac{\Gamma, \mu X \sigma \vdash \tau}{\mu l} \quad \frac{\Gamma, \tau \vdash \nu X \sigma}{\nu r} \quad \frac{\Gamma, \tau \vdash \nu X \sigma}{\nu l}
  \]

  \[
  \frac{\Gamma \vdash \mu X \sigma}{\mu r} \quad \frac{\Gamma, \mu X \sigma \vdash \tau}{\mu l} \quad \frac{\Gamma, \tau \vdash \nu X \sigma}{\nu r} \quad \frac{\Gamma, \tau \vdash \nu X \sigma}{\nu l}
  \]
The finitary proof system $\mu LJ$

- **Formulas:**

  \[ \sigma, \tau ::= X \mid 1 \mid \sigma \to \tau \mid \sigma \times \tau \mid \sigma + \tau \mid \mu X \sigma \mid \nu X \sigma \]

- **Condition:** $X$ occurs positively in $\sigma$ for $\mu X \sigma$ and $\nu X \sigma$.

- **Examples:** $B := 1 + 1$

  \[ \mu X (X \to B) \quad \mu X (B \to X) \quad \mu X ((X \to B) \to B) \]

- **Inference rules:**

  \[
  \begin{align*}
  \frac{\Gamma \vdash \sigma [\mu X \sigma / X]}{\Gamma \vdash \mu X \sigma} & \quad \mu r \\
  \frac{\Gamma, \sigma [\tau / X] \vdash \tau}{\Gamma, \mu X \sigma \vdash \tau} & \quad \mu l \\
  \frac{\Gamma, \tau \vdash \sigma [\tau / X]}{\Gamma, \tau \vdash \nu X \sigma} & \quad \nu r \\
  \frac{\Gamma, \sigma [\nu X \sigma / X] \vdash \tau}{\Gamma, \nu X \sigma \vdash \tau} & \quad \nu l
  \end{align*}
  \]
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  $$\sigma, \tau ::= X \mid 1 \mid \sigma \rightarrow \tau \mid \sigma \times \tau \mid \sigma + \tau \mid \mu X \sigma \mid \nu X \sigma$$

- **Condition:** $X$ occurs positively in $\sigma$ for $\mu X \sigma$ and $\nu X \sigma$.

- **Examples:** $B := 1 + 1$

  $$\mu X(X \rightarrow B) \quad \mu X(B \rightarrow X) \quad \mu X((X \rightarrow B) \rightarrow B)$$

- **Inference rules:**

  \[
  \begin{align*}
  \mu_r & \quad \frac{\Gamma \vdash \sigma[\mu X \sigma/X]}{\Gamma \vdash \mu X \sigma} \\
  \mu_l & \quad \frac{\Gamma, \mu X \sigma \vdash \tau}{\Gamma, \sigma[\tau/X] \vdash \tau} \\
  \nu_r & \quad \frac{\Gamma, \tau \vdash \sigma[\tau/X]}{\Gamma, \nu X \sigma \vdash \tau} \\
  \nu_l & \quad \frac{\Gamma, \nu X \sigma \vdash \tau}{\Gamma, \sigma[\nu X \sigma/X] \vdash \tau}
  \end{align*}
  \]
Some intuition

- Positivity condition on $\mu X\sigma(X)$ and $\nu X\sigma(X) \implies$ monotonicity

- Interpreting the inference rules:

  \[
  \sigma(\mu X\sigma) \rightarrow \mu X\sigma \\
  (\sigma(\tau) \rightarrow \tau) \rightarrow \mu X\sigma \rightarrow \tau
  \]

  $\mu X\sigma$ is a pre-fixed point

  $\mu X\sigma$ is the meet of all pre-fixed points

  \[
  \nu X\sigma \rightarrow \sigma(\nu X\sigma) \\
  (\tau \rightarrow \sigma(\tau)) \rightarrow \tau \rightarrow \nu X\sigma
  \]

  $\nu X\sigma$ is a post-fixed point

  $\nu X\sigma$ is the join of all post-fixed points

- Knaster-Tarski’s theorem:

  least fixed point $= \text{meet of all pre-fixed points}$

  greatest fixed point $= \text{join of all post-fixed points}$
Encodings inductive data types in $\mu$LJ

- **Natural numbers:** $n$ construed using $n + 1$ units

  \[
  \begin{align*}
  N & := \mu X (1 + X) & 0 & := 1_r \vdash 1 \\
  & \vdash 1 + N & n + 1 & := 1_r \vdash 1 + N \\
  & \vdash N & \mu r \vdash 1 + N & \vdash N
  \end{align*}
  \]

- **Addition:**

  \[
  \begin{align*}
  \text{add}(0, y) & = y \\
  \text{add}(x + 1, y) & = \text{add}(x, y) + 1
  \end{align*}
  \]

- **NB:** Representability via cut elimination.
Encodings inductive data types in $\mu$LJ

- **Natural numbers:** $n$ construed using $n + 1$ units

\[
N := \mu X(1 + X) \quad 0 := \begin{array}{c}
1_r \vdash 1 \\
\mu_r \vdash 1 + N \\
\mu_r \vdash N
\end{array} \quad n + 1 := \begin{array}{c}
1_r \vdash N \\
\mu_r \vdash 1 + N \\
\mu_r \vdash N
\end{array}
\]

- **Addition:**

\[
\text{add}(0, y) = y \\
\text{add}(x + 1, y) = \text{add}(x, y) + 1
\]

\[
\text{add} := \begin{array}{c}
\text{id} \quad N \vdash N \\
1_r \vdash N \\
\mu_l \vdash 1 + N, N \vdash N \\
\mu_l \vdash N, N \vdash N
\end{array}
\]

- **NB:** Representability via cut elimination.
Encodings inductive data types in $\mu LJ$

- **Natural numbers:** $n$ construed using $n + 1$ units

\[
N := \mu X(1 + X) \quad 0 := \begin{array}{ll}
1_r \vdash 1 \\
\mu_r \vdash 1 + N \\
\mu_r \vdash N
\end{array} \quad n + 1 := \begin{array}{ll}
+ \vdash 1 + N \\
\mu_r \vdash N
\end{array}
\]

- **Addition:**

\[
\begin{align*}
\text{add}(0, y) &= y \\
\text{add}(x + 1, y) &= \text{add}(x, y) + 1
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\]

\[
\text{add} := \begin{array}{ll}
\text{id} \vdash N \\
1_i \vdash 1, N \vdash N \\
\text{succ} \vdash N \vdash N \\
\mu_l \vdash 1 + N, N \vdash N \\
\mu_l \vdash N, N \vdash N
\end{array}
\]

- **NB:** Representability via cut elimination.
Encodings coinductive data types in $\mu$LJ

Streams of natural numbers:

$$S := \nu X(N \times X) \quad \text{hd} := \times I \frac{N \vdash N}{N \times S \vdash N} \nu I \frac{N \times S \vdash N}{S \vdash N}$$

$$tl := \times I \frac{S \vdash S}{N \times S \vdash S} \nu I \frac{N \times S \vdash S}{S \vdash N}$$

Unfolding: $n, \alpha, f \mapsto f(0) :: f(1) :: \ldots :: f(n-1) :: \alpha(n) :: \alpha(n+1) :: \ldots$
Encodings coinductive data types in $\mu$LJ

- Streams of natural numbers:

\[
S := \nu X (N \times X) \\
hd := \text{id} \quad \frac{N \vdash N}{N \times S \vdash N} \\
tl := \text{id} \quad \frac{N \times S \vdash S}{S \vdash N}
\]

- Unfolding: $n, \alpha, f \mapsto f(0) :: f(1) :: \ldots :: f(n-1) :: \alpha(n) :: \alpha(n+1) :: \ldots$

\[
\begin{align*}
\mu_l & \quad \frac{1 + (N \times S), S, N \vdash N \vdash N \times S}{1, S, N \vdash N \vdash N \times S} \\
1_l & \quad \frac{1, S, N \vdash N \vdash N \times S}{S, N \rightarrow N \vdash N \times S} \\
\nu_l & \quad \frac{S, N \rightarrow N \vdash N \times S}{S \vdash N} \\
\times_l & \quad \frac{N \times S \vdash S}{N \times S \vdash N} \\
\nu_r & \quad \frac{N, S, N \rightarrow N \vdash N \times S}{N, S, N \rightarrow N \vdash N \vdash S}
\end{align*}
\]
The circular proof system $C_{\mu LJ}$

- Non-wellfounded proofs generated by the inference rules:

\[
\begin{align*}
\frac{\mu^r}{\Gamma \vdash \sigma[\mu X \sigma/X]} & \quad \frac{\mu^l}{\Gamma, \sigma[\mu X \sigma/X] \vdash \tau} & \quad \frac{\nu^r}{\Gamma \vdash \sigma[\nu X \sigma/X]} & \quad \frac{\nu^l}{\Gamma, \nu X \sigma \vdash \tau}
\end{align*}
\]

- **Progressiveness** condition:
  - thread = maximal FL-subformula path along a branch
  - progressing proof = any infinite branch has a thread that is infinitely often principal and the smallest infinitely often principal formula is $\mu$-formula on the LHS or a $\nu$-formula on the RHS (of $\vdash$)
  - progressiveness allows to maintain logical consistency ($\nu \neq \mu$)

- **Regularity** condition:
  - circular proof = finitely many distinct subproofs
  - circular proofs admit finite presentation (finite trees with backpointers)

- **Circular system** $C_{\mu LJ} =$ circular and progressing non-wellfounded proofs.
The circular proof system $C_{\mu}LJ$

- Non-wellfounded proofs generated by the inference rules:

\[
\begin{align*}
\frac{\Gamma \vdash \sigma[\mu X \sigma / X]}{\mu_r} & \quad \frac{\Gamma, \sigma[\mu X \sigma / X] \vdash \tau}{\mu_I} & \quad \frac{\Gamma \vdash \sigma[\nu X \sigma / X]}{\nu_r} & \quad \frac{\Gamma, \sigma[\nu X \sigma / X] \vdash \tau}{\nu_I}
\end{align*}
\]

- **Progressiveness** condition:
  
  - thread = maximal FL-subformula path along a branch
  
  - progressing proof = any infinite branch has a thread that is infinitely often principal and the smallest infinitely often principal formula is $\mu$-formula on the LHS or a $\nu$-formula on the RHS (of $\vdash$)

  - progressiveness allows to maintain logical consistency ($\nu \neq \mu$)

- **Regularity** condition:
  
  - circular proof = finitely many distinct subproofs
  
  - circular proofs admit finite presentation (finite trees with backpointers)

- **Circular system** $C_{\mu}LJ = \text{circular and progressing non-wellfounded proofs.}$
The circular proof system \( C_{\mu LJ} \)

- Non-wellfounded proofs generated by the inference rules:

\[
\begin{align*}
\Gamma \vdash \sigma[\mu X\sigma / X] & \quad \mu_r \quad \Gamma, \sigma[\mu X\sigma / X] \vdash \tau & \quad \mu_l \quad \Gamma, \mu X\sigma \vdash \tau & \quad \nu_r \quad \Gamma \vdash \nu X\sigma & \quad \nu_l \quad \Gamma, \nu X\sigma \vdash \tau
\end{align*}
\]

- Progressiveness condition:
  - thread = maximal FL-subformula path along a branch
  - progressing proof = any infinite branch has a thread that is infinitely often principal and the smallest infinitely often principal formula is \( \mu \)-formula on the LHS or a \( \nu \)-formula on the RHS (of \( \vdash \))
  - progressiveness allows to maintain logical consistency \( (\nu \neq \mu) \)

- Regularity condition:
  - circular proof = finitely many distinct subproofs
  - circular proofs admit finite presentation (finite trees with backpointers)

- Circular system \( C_{\mu LJ} \) = circular and progressing non-wellfounded proofs.
The circular proof system \( C\mu LJ \)

- Non-wellfounded proofs generated by the inference rules:

\[
\begin{align*}
\Gamma \vdash \sigma[\mu X \sigma/X] & \quad \mu_r \quad \Gamma, \sigma[\mu X \sigma/X] \vdash \tau & \quad \mu_i \quad \Gamma, \mu X \sigma \vdash \tau \\
\Gamma, \sigma[\nu X \sigma/X] \vdash \tau & \quad \nu_r \quad \Gamma \vdash \nu X \sigma & \quad \nu_i \quad \Gamma, \nu X \sigma \vdash \tau 
\end{align*}
\]

- **Progressiveness** condition:
  - thread = maximal FL-subformula path along a branch
  - progressing proof = any infinite branch has a thread that is infinitely often principal and the smallest infinitely often principal formula is \( \mu \)-formula on the LHS or a \( \nu \)-formula on the RHS (of \( \vdash \))
  - progressiveness allows to maintain logical consistency (\( \nu \neq \mu \))

- **Regularity** condition:
  - circular proof = finitely many distinct subproofs
  - circular proofs admit finite presentation (finite trees with backpointers)

- **Circular system** \( C\mu LJ = \text{circular and progressing non-wellfounded proofs.} \)
Encodings inductive data types in $\text{C} \mu \text{LJ}$

- **Addition:**

  \[
  \text{add}(0, y) = y \\
  \text{add}(x + 1, y) = \text{add}(x, y) + 1
  \]

- **Streams of natural numbers:**

  \[
  n_0 :: n_1 :: n_2 \ldots := \\
  \frac{}{n_0} \quad \frac{n_1}{\nu_r} \\
  \frac{}{\nu_r} \\
  \frac{}{\nu_r} \\
  \frac{}{\nu_r}
  \]

- **Progressiveness condition is a totality criterion:**

  *termination* for inductive data \quad vs \quad *productivity* for coinductive data
Encodings inductive data types in $C\mu$LJ

- Addition:
  \[
  \begin{align*}
  \text{add}(0, y) &= y \\
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  \end{align*}
  \]

- Streams of natural numbers:
  \[
  n_0 :: n_1 :: n_2 :: \ldots ::
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  termination for inductive data vs productivity for coinductive data
Encodings inductive data types in $C_{\mu LJ}$

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- **Streams of natural numbers:**

  \[
  n_0 :: n_1 :: n_2 :: \ldots :: =
  \]

- **Progressiveness condition is a totality criterion:**

  termination for inductive data vs productivity for coinductive data
Proof systems

Theories of arithmetic

Overview of the results
The theory \( \mu PA \)

- **Formulas:** closure of \( \mathcal{L}_{PA} \) under fixed point operators

\[
\phi := t = u \mid t < u \mid t \in X \mid t \in \mu X \lambda x \phi \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \forall x \phi \mid \exists x \psi
\]

**Condition:** \( X \) occurs positively in \( \phi \) for \( t \in \mu X \lambda x \phi \).

- **Semantically:**
  - if \( \phi(X, x) \) then \( \lambda x \phi \) interpreted as \( \{n \in \mathbb{N} \mid \mathcal{N} \models \phi(n)\} \in \wp(\mathbb{N}) \)
  - This induces a **monotone** function:

\[
\wp(\mathbb{N}) \to \wp(\mathbb{N})
\]

\[
A \mapsto \{n \in \mathbb{N} \mid \mathcal{N} \models \phi(A, n)\}
\]

- \( \mu X. \lambda x. \phi \) interpreted as the least fixed point of the above function.

- **Theory \( \mu PA \)** = extension of PA by the following axioms for \( \phi(X, x), \psi(x) \):
  - \( \forall y (\phi(\mu X \lambda x \phi, y) \to y \in \mu X \lambda x \phi) \)
  - \( \forall x (\phi(\psi, x) \to \psi(x)) \to (y \in \mu X \lambda x \phi \to \psi(y)) \)
The theory $\mu PA$

- **Formulas**: closure of $L_{PA}$ under fixed point operators

\[
\phi := t = u \mid t < u \mid t \in X \mid t \in \mu X \lambda x \phi \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \forall x \phi \mid \exists x \psi
\]

**Condition**: $X$ occurs positively in $\phi$ for $t \in \mu X \lambda x \phi$.

- **Semantically**:
  - if $\phi(X, x)$ then $\lambda x \phi$ interpreted as $\{ n \in \mathbb{N} \mid \mathcal{N} \models \phi(n) \} \in \wp(\mathbb{N})$
  - This induces a **monotone** function:
    \[
    \wp(\mathbb{N}) \to \wp(\mathbb{N})
    \]
    \[
    A \mapsto \{ n \in \mathbb{N} \mid \mathcal{N} \models \phi(A, n) \}
    \]
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- **Theory $\mu PA$** = extension of PA by the following axioms for $\phi(X, x), \psi(x)$:
  - $\forall y(\phi(\mu X \lambda x \phi, y) \to y \in \mu X \lambda x \phi)$
  - $\forall x(\phi(\psi, x) \to \psi(x)) \to (y \in \mu X \lambda x \phi \to \psi(y))$
The theory $\mu PA$

- **Formulas:** closure of $\mathcal{L}_{PA}$ under fixed point operators

$$
\phi \; := \; t = u \mid t < u \mid t \in X \mid t \in \mu X \lambda x \phi \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \forall x \phi \mid \exists x \psi
$$

**Condition:** $X$ occurs positively in $\phi$ for $t \in \mu X \lambda x \phi$.

- **Semantically:**
  - if $\phi(X, x)$ then $\lambda x \phi$ interpreted as $\{n \in \mathbb{N} \mid \mathcal{N} \models \phi(n)\} \in \wp(\mathbb{N})$
  - This induces a **monotone** function:

\[
\wp(\mathbb{N}) \to \wp(\mathbb{N})
\]

\[A \mapsto \{n \in \mathbb{N} \mid \mathcal{N} \models \phi(A, n)\}\]

- $\mu X. \lambda x. \phi$ interpreted as the least fixed point of the above function.

- **Theory $\mu PA$** = extension of PA by the following axioms for $\phi(X, x), \psi(x)$:
  - $\forall y(\phi(\mu X \lambda x \phi, y) \rightarrow y \in \mu X \lambda x \phi)$
  - $\forall x(\phi(\psi, x) \rightarrow \psi(x)) \rightarrow (y \in \mu X \lambda x \phi \rightarrow \psi(y))$
The theory $\Pi^1_2$-CA$_0$ and conservativity properties

- $\Pi^1_2$-CA$_0 = \text{PA}_2$ with comprehension scheme restricted to $\Pi^1_2$ formulas:
  \[ \exists X \forall x (\phi(x) \leftrightarrow x \in X) \quad X \notin FV(\phi) \quad \phi \in \Pi^1_2 \]

- **Theorem [Mö02]**: $\Pi^1_2$-CA$_0$ is arithmetically conservative over $\mu\text{PA}$.

- **Theorem [Tup04]**: $\mu\text{PA}$ is $\Pi^0_2$-conservative over $\mu\text{HA}$.

- **Consequence**: $\mu\text{PA}$, $\mu\text{HA}$ and $\Pi^1_2$-CA$_0$ prove the totality of the same functions on natural numbers.
The theory $\Pi^1_2$-CA$_0$ and conservativity properties

- $\Pi^1_2$-CA$_0 = PA_2$ with comprehension scheme restricted to $\Pi^1_2$ formulas:

$$\exists X \forall x (\phi(x) \iff x \in X) \quad X \notin FV(\phi) \quad \phi \in \Pi^1_2$$

- **Theorem [Mö02]**: $\Pi^1_2$-CA$_0$ is arithmetically conservative over $\mu$PA.

- **Theorem [Tup04]**: $\mu$PA is $\Pi^0_2$-conservative over $\mu$HA.

- **Consequence**: $\mu$PA, $\mu$HA and $\Pi^1_2$-CA$_0$ prove the totality of the same functions on natural numbers.
Proof systems

Theories of arithmetic

Overview of the results
Back to our grand tour diagram

Theories of arithmetic:

$\muHA \leftarrow_{[\text{Tup04}]} \muPA \leftarrow_{[\text{Mö02}]} \Pi^1_2-\text{CA}_0$}

Proof systems:

$\muLJ \rightarrow_{\text{simulation}} C\muLJ \rightarrow_{\text{negative translation}} C\muLJ^-$

realisability

totality argument
Simulation of $\mu$LJ into $C\mu$LJ

**Idea:** compiling (co)iteration to loops

\[\frac{\Gamma, \sigma(\rho) \vdash \rho \quad \mu I}{\Gamma, \mu X\sigma(X) \vdash \rho} \quad \iff \quad \frac{\sigma}{\mu I \quad \Gamma, \sigma(\mu X\sigma(X)) \vdash \sigma(\rho) \quad \Gamma, \sigma(\rho) \vdash \rho}
\]

Dually for $\nu_r$. 
Theories of arithmetic:

- $\muHA$ [Tup04] $\muPA$ [Mö02] $\Pi_2^1-CA_0$

Proof systems:

- $\muLJ \downarrow$ simulation $\rightarrow C_{\muLJ}$ negative translation $\rightarrow C_{\muLJ^-}$

realisability

totality argument
Translation from $C\mu LJ$ to $C\mu LJ^−$

- $C\mu LJ^− = \{N, \times, \rightarrow, \mu\}$-fragment of $C\mu LJ$

- Kolmogorov-style negative translation with $\neg\sigma := \sigma \rightarrow N$ (Friedman-Dragalin):
  
  $\begin{align*}
  X^N &:= \neg\neg X \\
  1^N &:= \neg N \\
  (\sigma \times \tau)^N &:= \neg\neg(\sigma^N \times \tau^N) \\
  (\sigma \rightarrow \tau)^N &:= \neg\neg(\sigma^N \rightarrow \tau^N) \\
  (\sigma + \tau)^N &:= \neg(\neg\sigma^N \times \neg\tau^N) \\
  (\nu X\sigma)^N &:= \neg\neg\neg\mu X\neg\sigma^N[\neg X/X] \\
  (\mu X\sigma)^N &:= \neg\neg\mu X\sigma^N
  \end{align*}$

- **NB:** Negative translation simplifies our results (but we need non-strictly positive fixed points). Gödel-Gentzen-style translation does not preserve progressiveness.

- **Theorem:** $C\mu LJ$ and $C\mu LJ^−$ represent the same functions on natural numbers.
Proof systems

Theories of arithmetic:

\[ \mu \text{HA} \leftrightsquigarrow \mu \text{PA} \]

\[ \text{simulation} \quad \text{realisability} \quad \text{totality argument} \]

\[ \Pi^1_2-\text{CA}_0 \]

\[ \text{C}\mu\text{LJ}^\sim \]

\[ \mu \text{LJ} \]

\[ \text{C}\mu\text{LJ} \]

\[ [\text{Tup04}] \quad [\text{Mö02}] \]

realisability

negative translation
Rules as combinators, proofs as (co)terms

- Rules into combinators:

\[
\frac{\vec{\sigma}_1 \vdash \tau_1 \quad \vec{\sigma}_n \vdash \tau_n}{\vec{\sigma} \vdash \tau} \quad \leadsto \quad r : (\vec{\sigma}_1 \to \tau_1) \to \ldots \to (\vec{\sigma}_n \to \tau_n) \to \vec{\sigma} \to \tau
\]

- Circular proofs into (untyped) coterms:

\[
\mathcal{D} \in C_{\mu}LJ^- \quad \leadsto \quad t_{\mathcal{D}} \in \text{coTer}
\]

\[
\frac{\vec{\sigma}_1 \vdash \tau_1 \quad \ldots \quad \vec{\sigma}_n \vdash \tau_n}{\vec{\sigma} \vdash \tau} \quad \leadsto \quad r \ t_{\mathcal{D}_1} \ldots \ t_{\mathcal{D}_n}
\]

- Equational theory $=_{\text{cut}}$ on coterms essentially induced by cut elimination.
A totality argument for proofs of $C_{\mu LJ^-}$

- **Candidate** $A$ is subset of $\text{coTer}$ closed under $=_\text{cut}$.

- **Type structure**: with each formula $\sigma$ we associate a set $|\sigma| \subseteq \text{coTer}$ (the *domain of $\sigma$*):

  $$
  |A| := A \quad \text{A candidate}
  $$

  $$
  |N| := \{t \mid \exists n \in \mathbb{N}. t =_\text{cut} n\}
  $$

  $$
  |\sigma \rightarrow \tau| := \{t \mid \forall s \in |\sigma|. ts \in |\tau|\}
  $$

  $$
  |\mu X\sigma(X)| := \bigcap_{|\sigma(A)| \subseteq A} \quad \text{A candidate}
  $$

  $$
  |\sigma(\cdot)| := A \mapsto |\sigma(A)|
  $$

$\mu$-formulas as least fixed points: $|\mu X\sigma(X)| = \bigcup_{\alpha \in \text{Ord}} |\sigma^\alpha(\emptyset)|$. Important for logical complexity.

- **Theorem**: If $D$ is a circular proof of $\tau$ in $C_{\mu LJ^-}$ then $t_D \in |\tau|$.

- **Consequence**: $C_{\mu LJ^-}$ represents only total functions. We can formalise metamathematically the totality argument for $C_{\mu LJ^-}$ in $\Pi^1_2$-$\text{CA}_0$. 

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A totality argument for proofs of $C\mu LJ^-$

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- **Type structure:** with each formula $\sigma$ we associate a set $|\sigma| \subseteq \text{coTer}$ (the *domain of* $\sigma$):

  \[
  |A| := A \\
  |N| := \{ t \mid \exists n \in \mathbb{N}. t = \text{cut } n \} \\
  |\sigma \to \tau| := \{ t \mid \forall s \in |\sigma|. ts \in |\tau| \} \\
  |\mu X\sigma(X)| := \bigcap_{|\sigma(A)| \subseteq A} A \text{ candidate} \\
  |\sigma(\cdot)| := A \mapsto |\sigma(A)|
  \]

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- **Theorem:** If $D$ is a circular proof of $\tau$ in $C\mu LJ^-$ then $t_D \in |\tau|$.

- **Consequence:** $C\mu LJ^-$ represents only total functions. We can formalise metamathematically the totality argument for $C\mu LJ^-$ in $\Pi^1_2$-$\text{CA}_0$. 
A totality argument for proofs of $C_{\mu LJ^-}$

- **Candidate** $A$ is subset of coTer closed under $\mathcal{=}_{\text{cut}}$.

- **Type structure**: with each formula $\sigma$ we associate a set $|\sigma| \subseteq \text{coTer}$ (the *domain of $\sigma$*):

  - $|A| := A$  \hspace{2cm} $A$ candidate
  - $|N| := \{t \mid \exists n \in \mathbb{N}. t =_{\text{cut}} n\}$
  - $|\sigma \rightarrow \tau| := \{t \mid \forall s \in |\sigma|. ts \in |\tau|\}$
  - $|\mu X\sigma(X)| := \bigcap_{|\sigma(A)| \subseteq A} A$ candidate
  - $|\sigma(\cdot)| := A \mapsto |\sigma(A)|$

$\mu$-formulas as least fixed points: $|\mu X\sigma(X)| = \bigcup_{\alpha \in \text{Ord}} |\sigma^\alpha(\emptyset)|$. Important for logical complexity.

- **Theorem**: If $\mathcal{D}$ is a circular proof of $\tau$ in $C_{\mu LJ^-}$ then $t_\mathcal{D} \in |\tau|$.

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A totality argument for proofs of $C\mu LJ^-$

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- **Type structure**: with each formula $\sigma$ we associate a set $|\sigma| \subseteq \text{coTer}$ (the *domain of $\sigma$*):

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  |A| := A \quad \text{A candidate}
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  $$
  |N| := \{ t \mid \exists n \in \mathbb{N}. t =_{cut} n\}
  $$

  $$
  |\sigma \rightarrow \tau| := \{ t \mid \forall s \in |\sigma|. ts \in |\tau|\}
  $$

  $$
  |\mu X \sigma(X)| := \bigcap_{|\sigma(A)| \subseteq A} A \quad \text{A candidate}
  $$

  $$
  |\sigma(\cdot)| := A \mapsto |\sigma(A)|
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$\mu$-formulas as least fixed points: $|\mu X \sigma(X)| = \bigcup_{\alpha \in \text{Ord}} |\sigma^\alpha(\emptyset)|$. Important for logical complexity.

- **Theorem**: If $D$ is a circular proof of $\tau$ in $C\mu LJ^-$ then $t_D \in |\tau|$.

- **Consequence**: $C\mu LJ^-$ represents only total functions. We can formalise metamathematically the totality argument for $C\mu LJ^-$ in $\Pi^1_2$-CA$_0$. 

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Idea of the totality argument

**Theorem:** If $\forall \vec{s} \in |\vec{\sigma}|$, we have $t_D \vec{s} \in |\tau|$. 

**Idea of the proof.** By contradiction:
- Assume $t_D s_1 \ldots s_n \notin |\tau|$ for some $s_1 \in |\sigma_1|, \ldots, s_n \in |\sigma_n|$. If

$$D = \frac{D_1}{\vec{\sigma}_1 \vdash \tau_1} \ldots \frac{D_n}{\vec{\sigma}_n \vdash \tau_n}$$

with $\vec{s} \in |\vec{\sigma}|$ s.t. $t_D \vec{s} \notin |\tau|$, then there is a proof $D_i$ of $\vec{\sigma}_i \vdash \tau_i$ some inputs $\vec{s}_i \in |\vec{\sigma}_i|$ such that $t_{D_i} \vec{s}_i \notin |\tau_i|$. Construct infinite "non-total" branch $B$ in $D$.

- By progressiveness $B$ contains a thread (infinite FL-subformula path) where some $\mu X \sigma(X)$ (in negative position) unfolds infinitely often.

- Assign ordinals approximating the "critical" $\mu X \sigma(X)$, obtaining a non-increasing ordinal assignment along $B$. In particular, at any unfolding of the "critical" $\mu X \sigma(X)$ the corresponding ordinal strictly decreases. **Contradiction.**
Idea of the totality argument

**Theorem:** If \( \mathcal{D} \) in \( C\mu \text{LJ}^- \) then, for every \( \vec{s} \in |\vec{\sigma}| \), we have \( t_\mathcal{D} \vec{s} \in |\tau| \).

**Idea of the proof.** By contradiction:

- Assume \( t_\mathcal{D} s_1 \ldots s_n \not\in |\tau| \) for some \( s_1 \in |\sigma_1|, \ldots, s_n \in |\sigma_n| \). If

\[
\mathcal{D} = \begin{array}{c}
\vdash_D \\
\ldots \\
\vdash_D \\
\end{array}
\begin{array}{c}
\sigma_1 \vdash \tau_1 \\
\ldots \\
\sigma_n \vdash \tau_n \\
\end{array}
\]

with \( \vec{s} \in |\vec{\sigma}| \) s.t. \( t_\mathcal{D} \vec{s} \not\in |\tau| \), then there is a proof \( \mathcal{D}_i \) of \( \sigma_i \vdash \tau_i \) some inputs \( \vec{s}_i \in |\vec{\sigma}_i| \) such that \( t_{\mathcal{D}_i} \vec{s}_i \not\in |\tau_i| \). Construct infinite "non-total" branch \( B \) in \( \mathcal{D} \).

- By progressiveness \( B \) contains a thread (infinite FL-subformula path) where some \( \mu X \sigma(X) \) (in negative position) unfolds infinitely often.

- Assign ordinals approximating the "critical" \( \mu X \sigma(X) \), obtaining a non-increasing ordinal assignment along \( B \). In particular, at any unfolding of the "critical" \( \mu X \sigma(X) \) the corresponding ordinal strictly decreases. **Contradiction.**
Idea of the totality argument

- **Theorem:** If $\overline{\sigma} \vdash \tau$ in $C\mu$LJ$^-$ then, for every $\overline{s} \in |\overline{\sigma}|$, we have $t_D \overline{s} \in |\tau|$.

- **Idea of the proof.** By contradiction:
  
  - Assume $t_D s_1 \ldots s_n \not\in |\tau|$ for some $s_1 \in |\sigma_1|, \ldots, s_n \in |\sigma_n|$. If
    
    $\mathcal{D} = \begin{array}{c}
    \overline{\sigma}_1 \vdash \tau_1 \\
    \vdots \\
    \overline{\sigma}_n \vdash \tau_n \\
    \hline
    \overline{\sigma} \vdash \tau
    \end{array}$

    with $\overline{s} \in |\overline{\sigma}|$ s.t. $t_D \overline{s} \not\in |\tau|$, then there is a proof $\mathcal{D}_i$ of $\overline{\sigma}_i \vdash \tau_i$ some inputs $\overline{s}_i \in |\overline{\sigma}_i|$ such that $t_{D_i} \overline{s}_i \not\in |\tau_i|$. Construct infinite "non-total" branch $B$ in $\mathcal{D}$.

  - By **progressiveness** $B$ contains a thread (infinite FL-subformula path) where some $\mu X \sigma(X)$ (in negative position) unfolds infinitely often.

  - **Assign ordinals** approximating the "critical" $\mu X \sigma(X)$, obtaining a non-increasing ordinal assignment along $B$. In particular, at any unfolding of the "critical" $\mu X \sigma(X)$ the corresponding ordinal strictly decreases. **Contradiction.**
Idea of the totality argument

**Theorem:** If $\vec{\sigma} \vdash \tau$ in $C\mu LJ^-$ then, for every $\vec{s} \in |\vec{\sigma}|$, we have $t_D \vec{s} \in |\tau|$. 

**Idea of the proof.** By contradiction:

- Assume $t_D s_1 \ldots s_n \notin |\tau|$ for some $s_1 \in |\sigma_1|, \ldots, s_n \in |\sigma_n|$. If

$$D = \frac{D_1}{\vec{\sigma}_1 \vdash \tau_1} \ldots \frac{D_n}{\vec{\sigma}_n \vdash \tau_n} \frac{r}{\vec{\sigma} \vdash \tau}$$

with $\vec{s} \in |\vec{\sigma}|$ s.t. $t_D \vec{s} \notin |\tau|$, then there is a proof $D_i$ of $\vec{\sigma}_i \vdash \tau_i$ some inputs $\vec{s}_i \in |\vec{\sigma}_i|$ such that $t_{D_i} \vec{s}_i \notin |\tau_i|$. Construct infinite “non-total” branch $B$ in $D$.

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Theories of arithmetic:

\[ \mu \text{HA} \leftarrow [\text{Tup04}] \quad \mu \text{PA} \leftarrow [\text{Mö02}] \quad \Pi^1_2-\text{CA}_0 \]

Proof systems:

\[ \mu \text{LJ} \quad \mu \text{LJ} \]

realisability

- simulation

- negative translation

totality argument
Theories of arithmetic:

\[ \mu \text{HA} \leftarrow [\text{Tup04}] \quad \mu \text{PA} \leftarrow [\text{M"o02}] \quad \Pi^1_2-\text{CA}_0 \]

Proof systems:

\[ \mu \text{LJ} \quad \text{simulation} \quad C\mu \text{LJ} \quad \text{negative translation} \quad C\mu \text{LJ}^- \]

realisability

totality argument
From $\mu$HA to $\mu$LJ via realisability

- **Proofs as (untyped, inductive) terms:** $D \in \mu$LJ $\mapsto t_D \in \text{Ter}$

- For any $t \in \text{Ter}$ and $\phi(X_1, \ldots, X_n)$ formula we define:

\[
 t \mathbin{r} \phi \quad \text{w.r.t. realisability candidates } R_{X_1}, \ldots, R_{X_n} \subseteq \text{Ter} \times \mathbb{N}
\]

- $t \mathbin{r} (n \in X)$ if $t \mathbin{r} X \mathbin{n}$

- $t \mathbin{r} (n \in \mu X \lambda x \phi)$ if, for all $R_X$, $t \mathbin{r} \forall x (\phi(X, x) \rightarrow x \in X) \rightarrow n \in X$

**Idea.** $\mu$HA as a subsystem of $\text{HA}_2$: $t \in \mu X \lambda x \phi := \forall X (\forall x (\phi(X, x) \rightarrow x \in X) \rightarrow t \in X)$

**Theorem:** If $\mu$HA $\vdash \phi$ then there is $D$ in $\mu$LJ such that $t_D \mathbin{r} \phi$.

$\vdash t(\phi)$

**Consequence:** The provably recursive functions of $\mu$HA are representable in $\mu$LJ.
From $\mu$HA to $\mu$LJ via realisability

- Proofs as (untyped, inductive) terms: $D \in \mu$LJ $\mapsto$ $t_D \in \text{Ter}$

- For any $t \in \text{Ter}$ and $\phi(X_1, \ldots, X_n)$ formula we define:

$$t \models_\phi \text{ w.r.t. realisability candidates } R_{X_1}, \ldots, R_{X_n} \subseteq \text{Ter} \times \mathbb{N}$$

$$t \models_\phi (n \in X) \quad \text{if } t \models_{R_X} n$$

$$t \models_\phi (n \in \mu X \lambda x \phi) \quad \text{if, for all } R_X, \ t \models_{R_X} \forall x(\phi(X, x) \rightarrow x \in X) \rightarrow n \in X$$

**Idea.** $\mu$HA as a subsystem of HA$_2$:

$$t \in \mu X \lambda x \phi := \forall X(\forall x(\phi(X, x) \rightarrow x \in X) \rightarrow t \in X)$$

**Theorem:** If $\mu$HA $\vdash \phi$ then there is $\ulcorner \Delta \urcorner$ in $\mu$LJ such that $t_\Delta \models_\phi$.

$$\vdash t(\phi)$$

**Consequence:** The provably recursive functions of $\mu$HA are representable in $\mu$LJ
From $\mu$HA to $\mu$LJ via realizability

- Proofs as (untyped, inductive) terms: $D \in \mu$LJ $\mapsto t_D \in \text{Ter}$

- For any $t \in \text{Ter}$ and $\phi(X_1, \ldots, X_n)$ formula we define:

  \[ t \ r \phi \quad \text{w.r.t. realizability candidates } R_{X_1}, \ldots, R_{X_n} \subseteq \text{Ter} \times \mathbb{N} \]

  \[ t \ r (\bar{n} \in X) \quad \text{if } t \ R_X \ n \]

  \[ t \ r (\bar{n} \in \mu X \lambda x \phi) \quad \text{if, for all } R_X, \ t \ r \ \forall x(\phi(X, x) \rightarrow x \in X) \rightarrow n \in X \]

Idea. $\mu$HA as a subsystem of HA$_2$:

\[ t \in \mu X \lambda x \phi := \forall X(\forall x(\phi(X, x) \rightarrow x \in X) \rightarrow t \in X) \]

- **Theorem:** If $\mu$HA $\vdash \phi$ then there is $D$ in $\mu$LJ such that $t_D \ r \phi$.

  $\vdash t(\phi)$

- **Consequence:** The provably recursive functions of $\mu$HA are representable in $\mu$LJ
From $\mu$HA to $\mu$LJ via realisability

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- For any $t \in$Ter and $\phi(X_1, \ldots, X_n)$ formula we define:

$$t \triangleright_{\phi} \text{ w.r.t. realisability candidates } R_{X_1}, \ldots, R_{X_n} \subseteq \text{Ter} \times \mathbb{N}$$

$$t \triangleright (\overline{n} \in X) \quad \text{if } t \triangleright_X n$$

$$t \triangleright (\overline{n} \in \mu X \lambda x \phi) \quad \text{if, for all } R_X, \quad t \triangleright \forall x (\phi(X, x) \rightarrow x \in X) \rightarrow n \in X$$

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Extending our results to other fixed point logics

- The results extend to $\mu$MALL [Bae12] and its circular version $C_\mu$MALL:

\[
\begin{align*}
\muLJ & \leftarrow \text{this talk} & C_\muLJ \\
\downarrow \text{(*) translation} & & \uparrow \text{negative translation} \\
\muMALL & \rightarrow \text{simulation} & C_\muMALL
\end{align*}
\]

(*) Adapt standard translation $(\_)^\bullet : \LJ \rightarrow \LL$ with $(\sigma \rightarrow \tau)^\bullet = !\sigma^\bullet \rightarrow \tau^\bullet$ using encoding of exponentials [Bae12]:

\[
\begin{align*}
?\sigma & := \mu X(\bot \oplus (X \otimes X) \oplus \sigma) \\
!\sigma & := \nu X(1 \& (X \otimes X) \& \sigma)
\end{align*}
\]

- Similarly for $\mu$LL and $C_\mu$LL (fully fledged linear logic).