Weak Similarity in Higher-Order Mathematical Operational Semantics

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Abstract GSOS [Turi & Plotkin '97]

Operational rules

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

$$\widehat{=}$$

GSOS law

$$\underbrace{\varrho_X \colon \Sigma(X \times BX) \to B(\Sigma^* X)}_{\text{natural in } X \in \mathbb{C}}$$

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- **Syntax** functor $\Sigma\colon\ \mathbb{C} \to \mathbb{C}$ with free monad $\Sigma^*\colon\ \mathbb{C} \to \mathbb{C}$

  e.g. $\Sigma X = \underbrace{X \times X}_{||} + \cdots$ and $\Sigma^* X \ = \ \text{terms over } X$

- **Behaviour** functor $B\colon\ \mathbb{C} \to \mathbb{C}$

  e.g. $BX \ = \ (\mathcal{P}_{\mathsf{f}} X)^L$ (labelled transition systems)

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Operational rules $\;\widehat{=}\;$ GSOS law $\varrho_X \colon \Sigma(X \times BX) \to B(\Sigma^* X)$

Operational model

$$\begin{tikzcd}
\Sigma(\mu\Sigma) \arrow[r, "\cong"] & \mu\Sigma \arrow[r] & B(\mu\Sigma)
\end{tikzcd}$$

initial Σ-algebra (programs)

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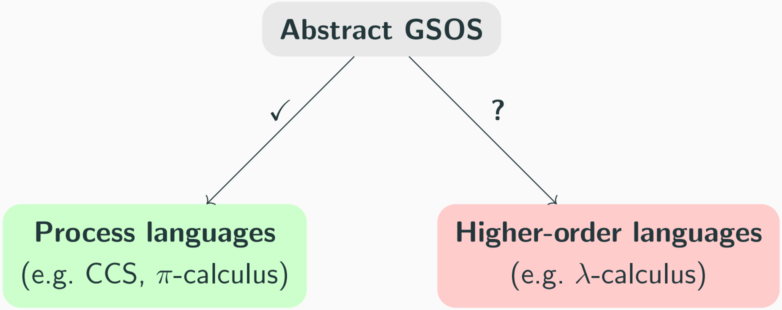
Theorem: Compositionality For Free!

Strong bisimilarity $\sim$ on the op. model $\mu\Sigma \to B(\mu\Sigma)$ is a congruence:

$$p_i \sim q_i \quad (i = 1, \dots, n) \quad \xRightarrow{f \in \Sigma} \quad \mathsf{f}(p_1, \dots, p_n) \sim \mathsf{f}(q_1, \dots, q_n).$$

Abstract GSOS for Higher-Order Languages?



Abstract GSOS

✓

?

Process languages
(e.g. CCS, π-calculus)

Higher-order languages
(e.g. λ-calculus)

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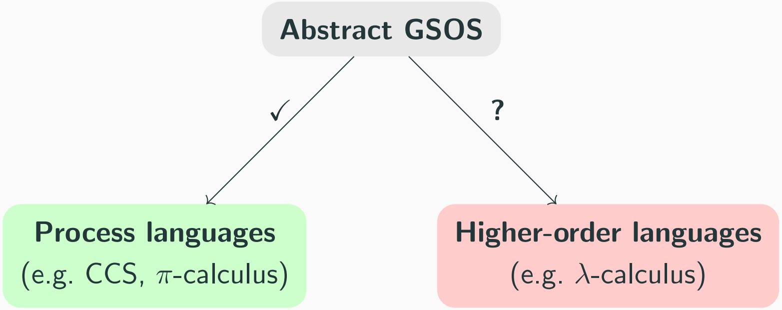

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$BX = X^X$ is not a functor

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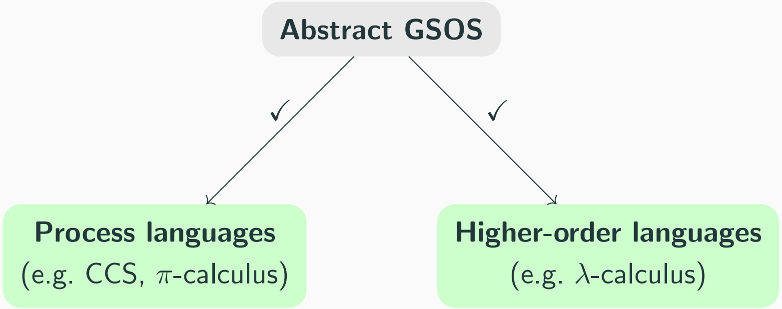

Abstract GSOS

✓ ✓

Process languages
(e.g. CCS, π-calculus)

Higher-order languages
(e.g. λ-calculus)

$BX = X^X$ is not a functor

…but $B(X, Y) = Y^X$ is a bifunctor.

Higher-Order Abstract GSOS [POPL'23]

GSOS law of $\Sigma\colon \mathbb{C} \to \mathbb{C}$ over $B\colon \mathbb{C} \to \mathbb{C}$ [Turi & Plotkin '97]

$$\underbrace{\varrho_X\colon \Sigma(X \times BX) \to B(\Sigma^* X)}_{\text{natural in } X \in \mathbb{C}}$$

$$\rightsquigarrow$$

Higher-order GSOS law of $\Sigma\colon \mathbb{C} \to \mathbb{C}$ over $B\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$

$$\underbrace{\varrho_{X,Y}\colon \Sigma(X \times B(X, Y)) \to B(X, \Sigma^*(X + Y))}_{\text{dinatural in } X \in \mathbb{C},\ \text{natural in } Y \in \mathbb{C}}$$

$$\widehat{=}$$

Operational rules of a higher-order language

Example: Call-by-Name λ-Calculus

Operational rules

$$\frac{}{(\lambda x.p)\,q \to p[q/x]}$$

$$\frac{p \to p'}{p\,q \to p'\,q}$$

$\widehat{=}$

Higher-order GSOS law

$$\varrho_{X,Y} \colon \Sigma(X \times B(X,Y)) \to B(X, \Sigma^*(X+Y))$$

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$$\varrho_{X,Y} \colon \Sigma(X \times B(X,Y)) \to B(X, \Sigma^*(X+Y))$$

$$\Sigma X = \underset{\text{variables}}{V} + \underset{\lambda\text{-abstraction}}{\delta X} + \underset{\text{application}}{X \times X} \quad \text{on} \quad \mathbb{C} = \mathbf{Set}^{\mathbb{F}}$$

$\cdots$ initial algebra $\mu\Sigma \in \mathbf{Set}^{\mathbb{F}}$, $\quad n \mapsto$ λ-terms in free vars from $\{x_1, \ldots, x_n\}$.

cf. Fiore, Plotkin & Turi ’99

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$$B(X,Y) = \underset{\text{substitution}}{\langle X, Y \rangle} \times (\underset{\text{reduction}}{Y} + \underset{\lambda x.(-)}{Y^X} + \underset{\text{stuck}}{1}) \quad \text{on} \quad \mathbb{C} = \mathbf{Set}^{\mathbb{F}}$$

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Higher-order operational rules $\widehat{=}$ Higher-order GSOS law $\varrho_{X,Y}$

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Compositionality for free?

Yes for strong bisimilarity! [POPL'23]

What about weak (bi-)similarity? [LICS'23]

Applicative similarity [Abramsky '90]

Weak coalgebraic similarity on operational model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$

$$=$$

Applicative similarity.

Example: Call-by-name λ-calculus

Greatest relation $\lesssim \subseteq \Lambda \times \Lambda$ such that for $t_1 \lesssim t_2$,

$$t_1 \to^\star \lambda x.t_1' \implies t_2 \to^\star \lambda x.t_2' \ \wedge \ \forall e \in \Lambda.\ t_1'[e/x] \lesssim t_2'[e/x].$$

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$$\underbrace{\textbf{Theorem [Abramsky].}\ \lesssim \text{ is a congruence.}}_{\textbf{Now: }\text{Generalization to higher-order abstract GSOS}}$$

Main Result

Compositionality Theorem for Higher-Order Abstract GSOS

Weak similarity on the op. model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence

if

the **weak operational model** is a **lax higher-order bialgebra**.

replace $\to$ by $\to^\star$

rules remain sound for weak transitions

$$\frac{p \to p'}{p\,q \to p'\,q} \quad \rightsquigarrow \quad \frac{p \to^\star p'}{p\,q \to^\star p'\,q}$$

cf. Bonchi, Petrișan, Pous, Rot ’15

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Howe's method

in abstract categories

cf. Dal Lago, Gavazzo, Levy '17

cf. Borthelle, Hirschowitz, Lafont '20

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Relation liftings

of behaviour bifunctors

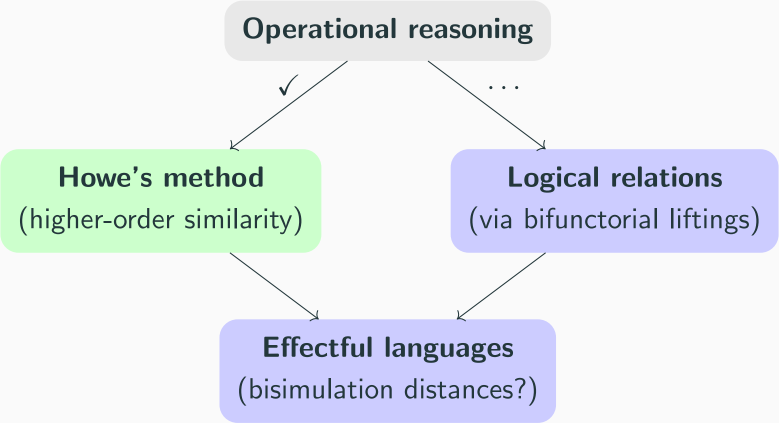
$$\lambda x.\ t \lesssim \lambda x.t' \Rightarrow \forall e \in \Lambda.t[e/x] \lesssim t'[e/x]$$

$$\lambda x.\ t \lesssim \lambda x.t' \Rightarrow \forall d \lesssim e.t[d/x] \lesssim t'[e/x]$$

$$\begin{tikzcd}
\mathbf{Rel}(\mathbb{C})^{\mathrm{op}} \times \mathbf{Rel}(\mathbb{C}) \arrow[r, "\overline{B}"] \arrow[d] & \mathbf{Rel}(\mathbb{C}) \arrow[d] \\
\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[r, "B"] & \mathbb{C}
\end{tikzcd}$$

Conclusion and Perspectives

Goal: Theory of HO languages at the level of higher-order abstract GSOS.



Operational reasoning

✓

…

Howe's method
(higher-order similarity)

Logical relations
(via bifunctorial liftings)

Effectful languages
(bisimulation distances?)