

The Size of Interpolants in Modal Logics

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Abstract

We start a systematic investigation of the size of Craig interpolants, uniform interpolants, and strongest implicates for (quasi-)normal modal logics. Our main upper bound states that for tabular modal logics, the computation of strongest implicates can be reduced in polynomial time to uniform interpolant computation in classical propositional logic. Hence they are of polynomial dag-size iff NP is included in P/poly. The reduction also holds for Craig interpolants if the tabular modal logic has the Craig interpolation property. Our main lower bound shows an unconditional exponential lower bound on the size of Craig interpolants and strongest implicates covering almost all non-tabular standard normal modal logics. For normal modal logics contained in or containing S4 or GL we obtain the following dichotomy: tabular logics have “propositionally sized” interpolants while for non-tabular logics an unconditional exponential lower bound holds.

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1 Overview

Craig interpolation and the stronger property of uniform interpolation have been investigated extensively for modal and other logics [32]. The Craig interpolation property (CIP) says that whenever an implication $\varphi \rightarrow \psi$ is valid there exists a formula χ using only the shared propositional variables of φ and ψ such that both $\varphi \rightarrow \chi$ and $\chi \rightarrow \psi$ are valid. The formula χ is then called a *Craig interpolant for $\varphi \rightarrow \psi$* and often provides a useful explanation of its validity [9]. Uniform interpolation (UIP) strengthens this. It says that for a signature σ of propositional variables, a Craig interpolant χ can be chosen uniformly for all ψ with $\text{sig}(\psi) \cap \text{sig}(\varphi) = \sigma$. Then χ is called a *uniform σ -interpolant of φ* and can be regarded as the result of eliminating existential quantifiers over, or “forgetting”, the variables $\text{sig}(\varphi) \setminus \sigma$ in φ [28, 34, 23, 33]. If a logic does not have CIP, *strongest σ -implicates*, formulas χ that behave like uniform σ -interpolants but in which the condition “ $\text{sig}(\psi) \cap \text{sig}(\varphi) = \sigma$ ” is replaced by “ $\text{sig}(\psi) \subseteq \sigma$ ” are often a sufficient approximation of uniform interpolants. They coincide with uniform interpolants if the logic has CIP.

While existence results and effective constructions for Craig interpolants, uniform interpolants, and strongest implicates in modal logics have been studied extensively, size bounds often remain unknown, even for standard modal logics. Moreover, in contrast to many other properties of modal logics such as the complexity of reasoning or the size of finite models, no general results have yet been established. In this paper we address the latter problem, and show first general results on the size of interpolants. Although we are mainly interested in normal modal logics, our results can often be more conveniently formulated for the even



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larger class of quasi-normal modal logics, modal logics determined by classes of (possibly generalised) Kripke frames with a distinguished world. We investigate modal logics with a single modal operator, but the extension to polymodal logics is straightforward.

The lack of general results in modal logic is not surprising: even in the case of propositional logic it is a major open problem whether Craig interpolants or uniform interpolants of polynomial size always exist. Although it is conjectured that this is false and both are of superpolynomial size in general, a proof would entail $P \neq NP$ and so is currently out of reach. As a result, the landscape of unconditional known lower bounds is rather sparse: currently, lower bounds have only been established for restricted fragments, most notably the monotone fragment of propositional logic, where one can show exponential lower bounds on the size of interpolants [29, 1]. Clearly, lower bounds obtained for propositional logic automatically transfer to modal logics. Thus, for example, the aforementioned exponential lower bound on the size of Craig interpolants in the monotone fragment of propositional logic immediately yields corresponding exponential lower bounds for monotone fragments of modal logics. An intriguing question, which we will pursue here, is when *upper bounds* for the case of propositional logic transfer to the modal setting.

Not every modal logic has the CIP, let alone the UIP. Indeed, while many standard modal logics have CIP, it is, for instance, known that in the uncountable set of normal modal logics containing S4 only at most 37 normal modal logics have CIP [12]. We refer the reader to Figure 1 for a small fragment of the landscape of modal logics, containing logics without CIP, logics with CIP and logics with UIP. Here and in what follows, a logic is called tabular if it is determined by a finite set of finite frames and pre-tabular if it is not tabular but all proper extensions are tabular. Note that a tabular logic with CIP also has UIP since one can obtain a uniform σ -interpolant for a formula φ by taking the conjunction over all relevant Craig interpolants (as there are only finitely many non-equivalent formulas over a finite signature of propositional variables).

We next discuss the main contributions of this paper. Unless stated otherwise, the size $|\varphi|$ of a formula φ is the length of φ if represented as a dag (or equivalently, the number of subformulas of φ). See also Section 7 for further discussion regarding dag-size and tree-size.

First contribution (strongest implicates and Craig interpolants for tabular logics). For every tabular modal logic L , the computation of strongest $L(\sigma)$ -implicates can be reduced in poly-time to the computation of uniform interpolants in propositional logic. As a corollary, we obtain that the size of strongest $L(\sigma)$ -implicates is bounded polynomially in the size of propositional uniform interpolants. Let $PL(\sigma)$ and $ML(\sigma)$ denote the set of propositional and, respectively, modal formulas using variables in σ only.

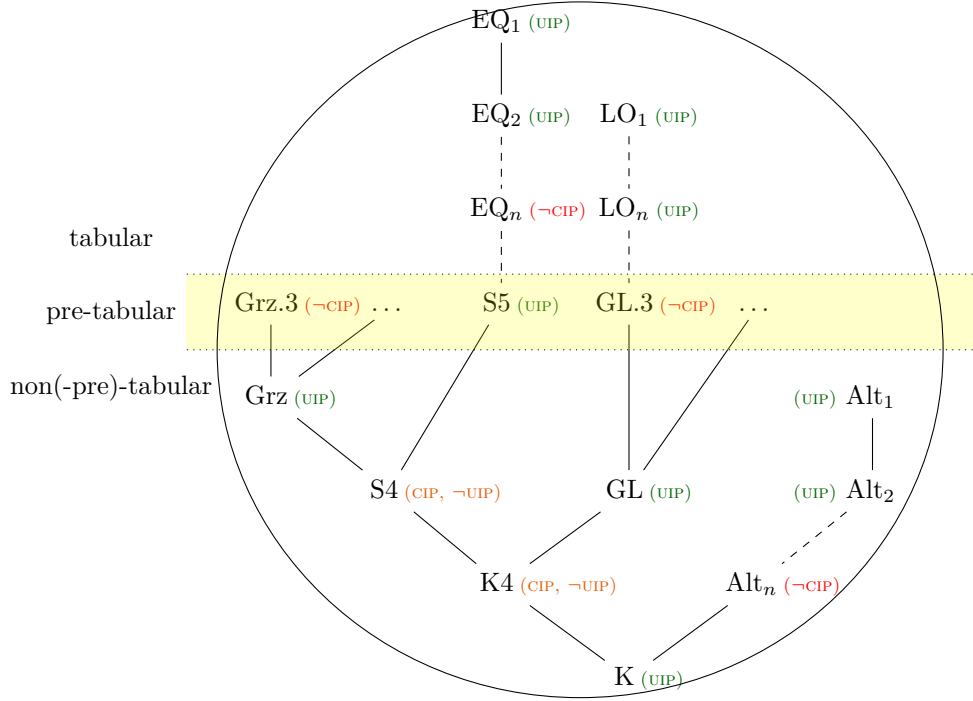
► **Theorem 1.1.** *Let L be a tabular quasi-normal modal logic. We can associate to each signature σ of propositional variables a companion signature $\hat{\sigma}$ such that there are poly-time translations*

$$tr_L : ML(\sigma) \rightarrow PL(\hat{\sigma}) \quad \text{and} \quad rt_L : PL(\hat{\sigma}) \rightarrow ML(\sigma)$$

such that for every $\varphi \in ML(\sigma)$, signature $\tau \subseteq sig(\varphi)$, and every uniform $\hat{\tau}$ -interpolant ψ of $tr_L(\varphi)$ in propositional logic, $rt_L(\psi)$ is a strongest $L(\tau)$ -implicate for φ .

We recall that quasi-normal modal logics form a generalization of normal modal logic. Hence, the above results (and the results below) apply in particular to normal modal logics.

► **Corollary 1.2.** *Let L be a tabular quasi-normal modal logic. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be any function such that every propositional formula φ has a uniform σ -interpolant of size at most $f(|\varphi|)$ for all $\sigma \subseteq sig(\varphi)$. Then every modal formula φ has a strongest $L(\sigma)$ -implicate (for $\sigma \subseteq sig(\varphi)$) of size at most $poly(f(poly(|\varphi|)))$.*



■ **Figure 1** Landscape of non-tabular, pre-tabular, and tabular normal modal logics. To keep the figure simple, not all inclusion relations that hold between the given logics are indicated with edges. The failure of CIP for EQ_n and for Alt_n holds for $n > 2$. Logics are defined formally in Section 2 and pointers to the literature and proofs are supplied in Section 3.

Similarly, for tabular modal logics L that have CIP, the computation of Craig interpolants in L can be reduced in poly-time to the computation of Craig interpolants in propositional logic. Consequently, the size of Craig interpolants in L is polynomial in the size of propositional Craig interpolants.

► **Theorem 1.3.** *Let L be a tabular quasi-normal modal logic that has CIP. Given a valid modal implication $\varphi \rightarrow \psi$ in L , and given a propositional Craig interpolant χ for $tr_L(\varphi) \rightarrow tr_L(\psi)$, it holds that $rt_L(\chi)$ is a Craig interpolant for $\varphi \rightarrow \psi$ with respect to L .*

Here, tr_L and rt_L refer to the poly-time translation from Theorem 1.1.

► **Corollary 1.4.** *Let L be a tabular quasi-normal modal logic that has CIP. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be any function such that every valid propositional implication $\varphi \rightarrow \psi$ has a Craig interpolant of size at most $f(|\varphi \rightarrow \psi|)$. Then every modal implication $\varphi \rightarrow \psi \in L$ has a Craig interpolant in L of size at most $poly(f(poly(|\varphi| + |\psi|)))$.*

The proof uses the standard bisimulation-based characterization of Craig interpolants and strongest implicates. A key ingredient of the proof is a poly-size representation of exponentially many non-bisimilar models using *abstract Φ -models*, for an appropriate poly-size set Φ of propositional formulas. Intuitively, an abstract Φ -model is a partial description of a Kripke model, which does not include a concrete propositional valuation for each world, but provides enough information to infer which worlds have the same propositional valuation.

Second contribution (exponential lower bounds for non-tabular logics). We prove a general exponential lower bound on the size of Craig interpolants and strongest implicates, which applies to many logics not covered by Theorem 1.1.

► **Theorem 1.5.** *Let L be any non-tabular normal modal logic that contains or is contained in $S4$ or the Gödel-Löb provability logic GL , or, more generally, has a non-tabular intersection with $S4$ or GL .*

1. *There exist modal formulas $(\varphi_n)_{n=1,2,\dots}$ of size polynomial in n and signatures $(\sigma_n)_{n=1,2,\dots}$ with $\sigma_n \subseteq \text{sig}(\varphi_n)$, such that φ_n has a strongest $L(\sigma_n)$ -implicate, and every strongest $L(\sigma_n)$ -implicate of φ_n has size at least 2^n .*
2. *There are modal formulas $(\varphi_n)_{n=1,2,\dots}$ and $(\psi_n)_{n=1,2,\dots}$ of size polynomial in n such that $\varphi_n \rightarrow \psi_n \in L$ for all $n \geq 1$, a Craig interpolant for $\varphi_n \rightarrow \psi_n$ exists for all $n \geq 1$, and every Craig interpolant for $\varphi_n \rightarrow \psi_n$ in L has size at least 2^n .*

The proof is based on a complete description of the pre-tabular normal modal logics containing $S4$ or GL and exploits their *poly-size model property*. In the case of strongest implicates, we actually prove a more general sufficient condition for an exponential lower bound that introduces the following *exponential growth property* of a modal logic: there are frames validating the logic in which one can reach exponentially many worlds in a polynomial number of steps from a distinguished world. All pre-tabular normal modal logics containing $S4$ or GL enjoy that property.

Interestingly, Theorem 1.1 and 1.3 and Theorem 1.5 together provide a dichotomy for normal modal logics L that are contained in or contain $S4$ or GL :

- If L is tabular, then L admits “propositional sized” strongest $L(\sigma)$ -implicates and (if L has CIP) Craig interpolants;
- If L is non-tabular, there are exponential lower bounds on the size of strongest $L(\sigma)$ -implicates and Craig interpolants.

Note that tabularity is decidable for normal modal logics containing $S4$ or GL , so this dichotomy is effective [8, Chapter 17].

We note that from Theorem 1.5 we obtain tight bounds for the size of Craig interpolants for many standard modal logics, including K , $K4$, $S4$, GL , Grzegorczyk’s logic Grz , and $S5$.

► **Theorem 1.6.** *Let L be either K , $K4$, $S4$, Grz , GL , or $S5$. Then for every $\varphi \rightarrow \psi \in L$ there exists a Craig interpolant of at most exponential size. This bound is optimal.*

The exponential upper bounds for K and $S5$ are shown in [6, 13] (even for uniform interpolants) and they follow from [21, 7] for the remaining logics. The exponential lower bound for GL was independently shown in [27], where exponential lower bounds for $K4$ and $S4$ are mentioned as open research problems.

Finally, we refute the natural conjecture that the above dichotomy for $S4$ and GL generalizes to all normal modal logics. We show that the non-tabular logic Alt_1 determined by frames in which every world has at most one successor admits a poly-time reduction of Craig and uniform interpolants to propositional Craig and, respectively, uniform interpolants.

The paper is structured as follows. In Section 2 we introduce the relevant notation and results in modal logic. Next, in Section 3, we discuss the main notions investigated in this paper and also review related work in propositional and modal logic. Section 4 is the core of this paper where we present the reduction of strongest implicates in tabular modal logics to uniform interpolants in propositional logic. The results and techniques presented in Section 4 are then also applied to give the reduction for Craig interpolants in tabular modal logics with CIP to propositional Craig interpolants in Section 5. Exponential lower bounds for interpolants are presented in Section 6. Details of some proofs are omitted. An extended version of the paper, which includes a technical appendix containing the proofs omitted here, can be found on arXiv.¹

¹ <https://arxiv.org/abs/2511.04577>

2 Preliminaries

We start by introducing relevant notions from modal and propositional logic [8, 5]. Fix a countable infinite set *Prop* of *atoms* or *propositional variables*. A *signature* σ is a finite subset of *Prop*. The language PL of *propositional formulas* consists of all formulas constructed from atoms using the propositional connectives $\top$, $\neg$, and $\wedge$ in the usual way. Similarly, the language ML of *modal formulas* consists of formulas constructed from atoms using the propositional connectives $\top$, $\neg$, and $\wedge$ as well as the modal operator $\Box$. We set $\Diamond\varphi = \neg\Box\neg\varphi$, define $\Box^{\leq n}\varphi = \varphi \wedge \Box\varphi \wedge \dots \wedge \Box^n\varphi$, and let $\Diamond^{\leq n}\varphi = \neg\Box^{\leq n}\neg\varphi$, for $n \geq 0$. By $\text{sig}(\varphi)$ we denote the set of atoms in the (propositional or modal) formula φ . For a given signature σ , $\text{PL}(\sigma)$ consists of those formulas $\varphi \in \text{PL}$ with $\text{sig}(\varphi) \subseteq \sigma$, and $\text{ML}(\sigma)$ again consists of those formulas $\varphi \in \text{ML}$ with $\text{sig}(\varphi) \subseteq \sigma$.

Models for propositional formulas are defined as usual. For a propositional formula φ , we write $\models \varphi$ if φ is valid. $\text{ML}(\sigma)$ is interpreted in *frames* $\mathfrak{F} = (W, R)$ with W a nonempty set of worlds and $R \subseteq W \times W$ an accessibility relation. A *pointed frame* is a pair $(\mathfrak{F}, w)$ with $w \in W$. A world $w \in W$ is called a *root* of $\mathfrak{F}$ if for every $v \in W$ there exists an R -path from w to v . A pointed frame $(\mathfrak{F}, w)$ with w a root of $\mathfrak{F}$ is called *rooted*. The *size* $|\mathfrak{F}|$ of a frame $\mathfrak{F}$ is the number of worlds in $\mathfrak{F}$.

A *model* based on $\mathfrak{F}$ is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ with $V : W \rightarrow 2^{\text{Prop}}$ a *valuation*. The *satisfaction-relation* $\models$ between *pointed models* $(\mathfrak{M}, w)$ with $w \in W$ and modal formulas φ is defined as follows:

- $\mathfrak{M}, w \models p$ if $p \in V(w)$;
- $\mathfrak{M}, w \models \neg\varphi$ if $\mathfrak{M}, w \not\models \varphi$;
- $\mathfrak{M}, w \models \varphi \wedge \psi$ if $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$;
- $\mathfrak{M}, w \models \Box\varphi$ if $\mathfrak{M}, v \models \varphi$ for all v with wRv .

We write $\mathfrak{F}, w \models \varphi$ and call φ *valid in* $(\mathfrak{F}, w)$ if $\mathfrak{M}, w \models \varphi$ for all models $\mathfrak{M}$ based on $\mathfrak{F}$. For $w \in W$ denote by $\mathfrak{F}_w$ the restriction of $\mathfrak{F}$ to the set of worlds $v \in W$ such that there is an R -path from w to v . Then w is a root of $\mathfrak{F}_w$ and one can easily show that $\mathfrak{F}, w \models \varphi$ iff $\mathfrak{F}_w, w \models \varphi$.

Let $\mathcal{F}$ be a set of pointed frames. Then

$$\text{Log}(\mathcal{F}) = \{\varphi \in \text{ML} \mid \text{for all } (\mathfrak{F}, w) \in \mathcal{F} : \mathfrak{F}, w \models \varphi\}$$

is the *logic determined by* $\mathcal{F}$. Note that $\text{Log}(\mathcal{F})$ is always a *quasi-normal modal logic*, a very general class of modal logics defined as follows [8]. The smallest quasi-normal modal logic is defined as $K = \text{Log}(\mathcal{F}_K)$, where $\mathcal{F}_K$ is the class of all pointed frames. Then a quasi-normal modal logic L is any set of modal formulas containing K and closed under modus ponens and uniform substitutions of atoms by modal formulas. We note that there are quasi-normal modal logics that are not of the form $\text{Log}(\mathcal{F})$ for any class $\mathcal{F}$ of pointed frames, but they will not play any important role in this paper. Observe that $\text{Log}(\mathcal{F}) = \text{Log}(\{(\mathfrak{F}_w, w) \mid (\mathfrak{F}, w) \in \mathcal{F}\})$, so we can always work with rooted frames instead of arbitrary pointed frames. If $L = \text{Log}(\mathcal{F})$ for some class $\mathcal{F}$ such that for every $(\mathfrak{F}, w) \in \mathcal{F}$ the set $\mathcal{F}$ also contains all $(\mathfrak{F}_v, v)$ with v in $\mathfrak{F}$, then $\text{Log}(\mathcal{F})$ is a *normal modal logic*, a quasi-normal modal logic that is also closed under the rule: if $\varphi \in L$, then $\Box\varphi \in L$. For normal modal logics L we often drop the distinguished worlds when defining L as $\text{Log}(\mathcal{F})$. Table 1 defines relevant normal modal logics.

We refer the reader to [8] for a detailed discussion of (quasi-) normal modal logics. A quasi-normal modal logic L is *tabular* if there exists a finite set $\mathcal{F}$ of finite pointed frames such that $L = \text{Log}(\mathcal{F})$.

■ **Table 1** List of relevant normal modal logics.

$\text{Log}(\mathcal{F})$	Class $\mathcal{F}$ of frames $\mathfrak{F} = (W, R)$
K	all frames
Alt_n	frames in which each node has at most n successors
S4	transitive and reflexive frames
Grz	frames validating S4 and no infinite strict chains $w_0 R w_1 R w_2 \dots$
GL	transitive and no infinite chains $w_0 R w_1 R w_2 \dots$
S5	equivalence relations
Grz.3/GL.3	GL/Grz frames satisfying $\forall w \forall v (w R v \vee v R w \vee w = v)$
LO_n	GL.3 frames of length at most n
EQ_n	S5 frames of size at most n

Let Φ be a set of formulas and let $\mathfrak{M}_i = (\mathfrak{F}_i, V_i)$ be models with $\mathfrak{F}_i = (W_i, R_i)$, for $i = 1, 2$. Then $\mathfrak{M}_1, w_1$ and $\mathfrak{M}_2, w_2$ are Φ -indistinguishable, in symbols $\mathfrak{M}_1, w_1 \equiv_{\Phi} \mathfrak{M}_2, w_2$, if $\mathfrak{M}_1, w_1$ and $\mathfrak{M}_2, w_2$ satisfy the same formulas in Φ . Let σ be a signature. Then $\text{ML}(\sigma)$ -indistinguishability is characterized using σ -bisimulations. A σ -bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is any relation $Z \subseteq W_1 \times W_2$ such that for all $w_1 \in W_1$ and $w_2 \in W_2$:

Atoms if $w_1 Z w_2$, then $\mathfrak{M}_1, w_1 \models p$ iff $\mathfrak{M}_2, w_2 \models p$ for all $p \in \sigma$,

Forth if $w_1 Z w_2$ and $(w_1, v_1) \in R_1$, then there is a $v_2 \in W_2$ such that $(w_2, v_2) \in R_2$ and $v_1 Z v_2$,

Back if $w_1 Z w_2$ and $(w_2, v_2) \in R_2$, then there is a $v_1 \in W_1$ such that $(w_1, v_1) \in R_1$ and $v_1 Z v_2$.

If Z is a σ -bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that $w_1 Z w_2$, we write $Z : \mathfrak{M}_1, w_1 \sim_{\sigma} \mathfrak{M}_2, w_2$. If there is any Z such that $Z : \mathfrak{M}_1, w_1 \sim_{\sigma} \mathfrak{M}_2, w_2$ we write $\mathfrak{M}_1, w_1 \sim_{\sigma} \mathfrak{M}_2, w_2$ and call $\mathfrak{M}_1, w_1$ and $\mathfrak{M}_2, w_2$ σ -bisimilar. It is known that $\sim_{\sigma}$ and $\equiv_{\text{ML}(\sigma)}$ coincide in finite pointed models [5].

3 Main Notions

In this paper, our main concern is the size and computation of strongest σ -implicates, uniform interpolants, and Craig interpolants. We next introduce these notions in detail in both modal and propositional logic.

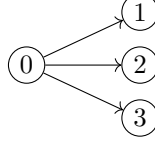
Consider a quasi-normal logic L , a modal formula φ and a signature $\sigma \subseteq \text{sig}(\varphi)$. A modal formula χ is called a *strongest $L(\sigma)$ -implicate* of φ if $\text{sig}(\chi) = \sigma$ and

- $\varphi \rightarrow \chi \in L$;
- if $\varphi \rightarrow \psi \in L$ and $\text{sig}(\psi) \subseteq \sigma$, then $\chi \rightarrow \psi \in L$.

By definition, any two strongest $L(\sigma)$ -implicates of φ are logically equivalent in L , so we speak about *the* strongest $L(\sigma)$ -implicate if it exists. In this paper, we are interested in the computation and size of strongest $L(\sigma)$ -implicates. There is a close relationship between strongest implicates and uniform interpolants. Recall that a formula χ is called a *Craig interpolant in L* for an implication $\varphi \rightarrow \psi \in L$ if $\varphi \rightarrow \chi \in L$, $\chi \rightarrow \psi \in L$, and $\text{sig}(\chi) \subseteq \text{sig}(\varphi) \cap \text{sig}(\psi)$. A logic L has the *Craig interpolation property (CIP)* if for any $\varphi \rightarrow \psi \in L$ a Craig interpolant exists. A formula χ is called a *uniform $L(\sigma)$ -interpolant* for φ if it satisfies the conditions of a strongest $L(\sigma)$ -implicate of φ even if Condition 2 is strengthened by replacing the condition “ $\text{sig}(\psi) \subseteq \sigma$ ” by the condition $\text{sig}(\psi) \cap \text{sig}(\varphi) \subseteq \sigma$. It follows that a uniform $L(\sigma)$ -interpolant for φ is a Craig interpolant for all $\varphi \rightarrow \psi \in L$ with $\text{sig}(\psi) \cap \text{sig}(\varphi) \subseteq \sigma$. The following is straightforward.

► **Lemma 3.1.** *If L has CIP, then strongest $L(\sigma)$ -implicates and uniform $L(\sigma)$ -interpolants coincide: if χ is a strongest $L(\sigma)$ -implicate of φ , then it is a uniform $L(\sigma)$ -interpolant for φ .*

► **Example 3.2.** Let $L_{\mathfrak{G}} = \text{Log}(\{(\mathfrak{G}, 0)\})$ with $\mathfrak{G}$ the frame below:



The logic $L_{\mathfrak{G}}$ does not have the CIP. For example, the strongest $L_{\mathfrak{G}}(\{p\})$ -implicate of $\Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q)$ is $\Diamond p$. Yet $\Diamond p$ is not a uniform $L_{\mathfrak{G}}(\{p\})$ -interpolant, since $(\Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q)) \rightarrow (\Diamond(\neg p \wedge r) \rightarrow \Box(\neg p \rightarrow r)) \in L_{\mathfrak{G}}$ while $\Diamond p \rightarrow (\Diamond(\neg p \wedge r) \rightarrow \Box(\neg p \rightarrow r)) \notin L_{\mathfrak{G}}$.

The following result characterizes strongest $L(\sigma)$ -implicates using bisimulations: a strongest $L(\sigma)$ -implicate of φ expresses existentially quantifying $\text{sig}(\varphi) \setminus \sigma$ modulo a σ -bisimulation. Similar characterisations have been shown for many modal logics [28, 34, 10, 24].

► **Theorem 3.3.** *Let $L = \text{Log}(\mathcal{F}_L)$ with $\mathcal{F}_L$ a finite sets of finite pointed frames. Let φ be a modal formula and $\sigma \subseteq \text{sig}(\varphi)$. Then the following conditions are equivalent for all modal formulas χ with $\text{sig}(\chi) \subseteq \sigma$:*

1. χ is the strongest $L(\sigma)$ -implicate of φ ;
2. for all models $(\mathfrak{M}, w)$ based on some $(\mathfrak{F}, w) \in \mathcal{F}_L$: $\mathfrak{M}, w \models \chi$ iff there exists a model $(\mathfrak{M}', w')$ based on some $(\mathfrak{F}', w') \in \mathcal{F}_L$ with $\mathfrak{M}', w' \sim_{\sigma} \mathfrak{M}, w$ and $\mathfrak{M}', w' \models \varphi$.

The definitions of strongest σ -implicates, uniform interpolants, and Craig interpolants in propositional logic are derived from the definitions for modal logic in the obvious way. As propositional logic has CIP, strongest propositional σ -implicates are uniform propositional σ -interpolants.

Define the *dag-size* $|\varphi|$ of a propositional or modal formula φ as the number of its distinct subformulas. This corresponds to representing formulas as directed acyclic graphs and is the standard measure used in circuit complexity theory. If we take into account the number of different occurrences of subformulas, we obtain the *tree-size* $s(\varphi)$ of φ . Clearly, $s(\varphi) \geq |\varphi|$ and it is conjectured that in propositional logic there is a superpolynomial (even exponential) gap between dag-size and tree-size. However, this conjecture remains open [30].

It is also conjectured that both propositional Craig interpolants and uniform interpolants are (in the worst case) necessarily of exponential dag-size. In more detail, the following result shows how the size of propositional interpolants is related to open problems in circuit complexity [20], see also [26, 31].

► **Theorem 3.4.**

- (1) *Propositional logic has poly-dag-size uniform interpolants if, and only if, $\text{NP} \subseteq \text{P/poly}$.*
- (2) *If propositional logic has poly-dag-size Craig interpolants, then $\text{NP} \cap \text{coNP} \subseteq \text{P/poly}$.*

We remind the reader that the question whether $\text{NP} \subseteq \text{P/poly}$ is a long-standing open problem. It is unlikely that it holds since, by the Karp-Lipton Theorem, if $\text{NP} \subseteq \text{P/poly}$, then the polynomial hierarchy collapses at the second level [19, 18, 2]. Conversely, if we could prove $\text{NP} \not\subseteq \text{P/poly}$, then $\text{NP} \neq \text{P}$ would follow since $\text{P} \subseteq \text{P/poly}$. It is also regarded as unlikely that $\text{NP} \cap \text{coNP} \subseteq \text{P/poly}$. However, if we could prove $\text{NP} \cap \text{coNP} \not\subseteq \text{P/poly}$, then again $\text{NP} \neq \text{P}$ would follow.

We next review relevant results on interpolants in modal logic. Among the main modal logics considered in this paper, K, S5, GL, and Grz have UIP [34] while S4 and K4 do not [14, 4]. S4 and K4 have CIP, however, while GL.3 and Grz.3 do not even enjoy CIP [12].

Regarding the size of Craig/uniform interpolants for the logics with CIP and UIP considered in this paper, for modal logics K and S5 it is known that uniform interpolants of at most exponential size always exist [6, 13]. In contrast, for GL and Grz, to date, only non-elementary methods are known for constructing uniform interpolants. Interestingly, however, there are tools available for computing uniform interpolants in these logics [11]. For Craig interpolants, for K and S5, the exponential upper bound is inherited from the exponential upper bound for uniform interpolants. For K4, S4, Grz, and GL one can obtain exponential upper bounds by using the reductions to Craig interpolants in K given in [21, Section 3.7] and applying the tableau developed in [7]. While this is done explicitly only for K4, it is straightforward to extend the proof to S4, Grz, and GL.

It is also worth noting that it is undecidable whether a finitely axiomatized normal modal logic has the CIP [21, Section 9.6]. By the same technique, one can show that the UIP is likewise undecidable. However, this does not preclude the possibility of establishing effective dichotomies for the size of interpolants. Indeed, even though CIP is undecidable for finitely axiomatized normal modal logics containing GL [8, Theorem 17.19], we are nonetheless able to prove a (conditional) dichotomy result for interpolant size for normal modal logics containing GL. This dichotomy is effective since it is determined by tabularity which is effective for finitely axiomatized normal modal logics containing GL.

4 Constructing Strongest Implicates via Propositional Logic

We show how one can construct, for tabular quasi-normal logics, in polynomial time strongest implicates from uniform interpolants in propositional logic.

4.1 Abstraction

The number of different valuations V of atoms in a signature σ in a fixed finite frame is exponential in $|\sigma|$, even for frames with a single world. To avoid working with all valuations, in this subsection we introduce abstract models, of which we only require polynomially many to supply sufficient information about the underpinning models for our purposes. Unless stated otherwise, in this section, we only consider finite rooted pointed frames $(\mathfrak{F}, w)$. We typically denote the root w by 0.

Given a signature σ , we call a finite set $\Phi \subseteq \text{PL}(\sigma)$ an *exhaustive and mutually exclusive set over σ* , σ -EME for short, if (i) $\models \bigvee_{\varphi \in \Phi} \varphi$ and (ii) $\models \neg(\varphi \wedge \psi)$ for all $\varphi \neq \psi$ with $\varphi, \psi \in \Phi$. If $\mathfrak{M} = (\mathfrak{F}, V)$ is a model and Φ a σ -EME, we say that Φ is a σ -cover of $\mathfrak{M}$ if for all $\varphi \in \Phi$ and all $w_1, w_2 \in W$: if $\mathfrak{M}, w_1 \models \varphi$ and $\mathfrak{M}, w_2 \models \varphi$ then $\mathfrak{M}, w_1 \equiv_{\text{PL}(\sigma)} \mathfrak{M}, w_2$.

If Φ is a σ -EME, then every world of $\mathfrak{M}$ satisfies exactly one formula $\varphi \in \Phi$. So Φ is a σ -cover of $\mathfrak{M}$ if, and only if, any two worlds of $\mathfrak{M}$ that are distinguishable in $\text{PL}(\sigma)$ are distinguished by Φ .

► **Lemma 4.1.** *For every σ -EME Φ and for every rooted model $\mathfrak{M}, 0$ of size at most N , we have Φ is a σ -cover of $\mathfrak{M}$ if, and only if, $\mathfrak{M}, 0 \models \text{cover}(\sigma, \Phi)$ where*

$$\text{cover}(\sigma, \Phi) = \bigwedge_{\varphi \in \Phi} \bigwedge_{p \in \sigma} (\Diamond^{\leq N}(\varphi \wedge p) \rightarrow \Box^{\leq N}(\varphi \rightarrow p))$$

The proof is straightforward, so we omit it here.

A key observation that we will make use of is that, even though the number of possible σ -valuations over a given frame is exponential in $|\sigma|$, the number of σ -EMEs needed in order to cover all of them is only polynomial in $|\sigma|$.

► **Lemma 4.2.** *Fix a finite frame $\mathfrak{F} = (W, R)$. For every signature σ , we can compute in polynomial time a set $\Gamma_{\mathfrak{F}}$ of σ -EMEs such that every model based on $\mathfrak{F}$ has a σ -cover in $\Gamma_{\mathfrak{F}}$. Moreover, each $\Phi \in \Gamma_{\mathfrak{F}}$ consists of at most $|W|$ formulas, each of size $O(|W|)$.*

We omit details of the proof, a sketch of the proof strategy is as follows. In any model based on $\mathfrak{F}$, there are at most $|W|$ distinguishable worlds. Using a partition refinement procedure we can find, for each indistinguishability class, an identifying formula that is a conjunction of literals over σ of size at most $O(|W|)$. Hence every model has a σ -cover Φ such that $|\Phi| \leq |W|$ and $|\varphi| \leq O(|W|)$ for each $\varphi \in \Phi$. Because the bounds on $|\Phi|$ and $|\varphi|$ do not depend on σ , each such cover contains only a constant number of atoms. There are polynomially many ways to choose a constant number of atoms from σ , so $\Gamma_{\mathfrak{F}}$ can be computed in polynomial time (and $|\Gamma_{\mathfrak{F}}|$ is polynomial).

► **Example 4.3.** Assume $\sigma = \{p, q\}$. If $\mathfrak{F}$ has a single world, then we can take $\Gamma_{\mathfrak{F}} = \{\{\top\}\}$ and if $\mathfrak{F}$ has two worlds, $\Gamma_{\mathfrak{F}} = \{\{p, \neg p\}, \{q, \neg q\}, \{\top\}\}$ is as required.

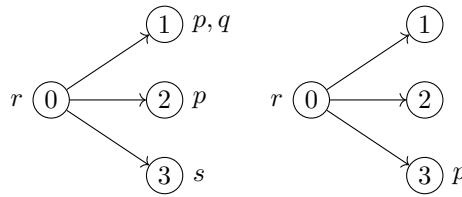
If Φ is a σ -EME, then an *abstract Φ -model* is a pair $\mathfrak{A} = (\mathfrak{F}, f)$, where $\mathfrak{F} = (W, R)$ is a frame and $f : W \rightarrow \Phi$ is a function. One can regard abstract Φ -models as partially specified models: rather than stating for every world w and atom p whether p is true at w , they only state which element of Φ is true in w . If Φ is a σ -cover of $\mathfrak{M}$, then the *Φ -abstraction of $\mathfrak{M}$* is the unique abstract Φ -model $\mathfrak{A} = (\mathfrak{F}, f)$ such that $\mathfrak{M}, w \models f(w)$ for all $w \in W$.

We also lift bisimulations from the level of models to the level of abstract Φ -models. Let Φ be a σ -EME and $\mathfrak{A}_i = (\mathfrak{F}_i, f_i)$ be abstract Φ -models, for $i = 1, 2$. A relation $Z \subseteq W_1 \times W_2$ is an *abstract Φ -bisimulation* between $\mathfrak{A}_1$ and $\mathfrak{A}_2$ if conditions **Forth** and **Back** introduced in the definition of σ -bisimulations hold and

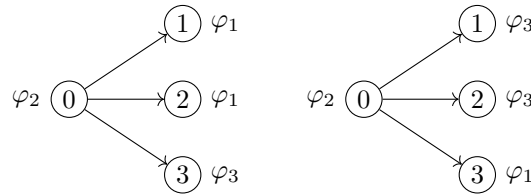
Abstract atoms for all $w_1 \in W_1$ and $w_2 \in W_2$, if $w_1 Z w_2$, then $f_1(w_1) = f_2(w_2)$.

We write $Z : \mathfrak{A}_1, w_1 \sim_{\Phi} \mathfrak{A}_2, w_2$ if Z is an abstract Φ -bisimulation such that $w_1 Z w_2$. If there is a Z such that $Z : \mathfrak{A}_1, w_1 \sim_{\Phi} \mathfrak{A}_2, w_2$ we write $\mathfrak{A}_1, w_1 \sim_{\Phi} \mathfrak{A}_2, w_2$ and say that $\mathfrak{A}_1, w_1$ and $\mathfrak{A}_2, w_2$ are *abstractly Φ -bisimilar*.

► **Example 4.4.** Consider the following two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$, based on the frame $\mathfrak{G}$ from Example 3.2.



Let $\sigma = \{p, r, s\}$, $\varphi_1 = p$, $\varphi_2 = \neg p \wedge r$ and $\varphi_3 = \neg p \wedge \neg r$. Then $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ is a σ -cover of both $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Furthermore, their Φ -abstractions, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, are shown below.



The models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not σ -bisimilar, but their abstractions $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are abstractly Φ -bisimilar, as witnessed by $Z = \{(0, 0), (1, 3), (2, 3), (3, 1), (3, 2)\}$.

While Z is not a σ -bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, we will later rely on the fact that there is a different model $\mathfrak{M}'_2$ such that Z is a σ -bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}'_2$.

► **Proposition 4.5.** *Let $\mathfrak{M}, 0$ be a rooted model, Φ a σ -cover of $\mathfrak{M}$ and $\mathfrak{A}$ the Φ -abstraction of $\mathfrak{M}$. Then for every rooted frame $(\mathfrak{F}', 0)$ with $\mathfrak{F}' = (W', R')$ and for every $Z \subseteq W \times W'$ the following are equivalent:*

1. $Z : (\mathfrak{M}, 0) \sim_\sigma (\mathfrak{M}', 0)$ for some model $\mathfrak{M}'$ based on $\mathfrak{F}'$,
2. $Z : (\mathfrak{A}, 0) \sim_\Phi (\mathfrak{A}', 0)$ for some abstract Φ -model $\mathfrak{A}'$ based on $\mathfrak{F}'$.

Furthermore, if such $\mathfrak{A}'$ and $\mathfrak{M}'$ exist then $\mathfrak{A}'$ is the Φ -abstraction of $\mathfrak{M}'$.

The proof is straightforward, we omit it here.

For a σ -EME Φ , we often treat the propositional formulas in Φ as if they are atoms. Specifically, we will denote by $\text{ML}(\Phi)$ the set of modal formulas constructed from the formulas in Φ (treated as atoms) in the usual way. In abstract Φ -models $\mathfrak{A}$, formulas $\psi \in \text{ML}(\Phi)$ are evaluated in the expected way by setting $\mathfrak{A}, w \models \varphi$ iff $f(w) = \varphi$ for all $\varphi \in \Phi$ and then using the standard truth conditions for modal formulas. Observe that if $\psi \in \text{ML}(\Phi)$ and $\mathfrak{A}$ is the Φ -abstraction of $\mathfrak{M}$, then $\mathfrak{M}, w \models \psi$ if, and only if, $\mathfrak{A}, w \models \psi$.

► **Proposition 4.6.** *Fix a natural number $N > 0$. Given a σ -EME Φ (with σ a signature) and a rooted abstract Φ -model $(\mathfrak{A}, 0)$ of size at most N , we can compute in polynomial time a formula $\delta_{\mathfrak{A}} \in \text{ML}(\Phi)$ such that, for all rooted abstract Φ -models $(\mathfrak{A}', 0)$ of size at most N , $\mathfrak{A}', 0 \models \delta_{\mathfrak{A}}$ iff $\mathfrak{A}', 0 \sim_\Phi \mathfrak{A}, 0$. Moreover, $|\delta_{\mathfrak{A}}| \leq O(2^N \cdot \max_{\varphi \in \Phi} |\varphi|)$.*

We omit some details of the proof, a sketch of the proof strategy is as follows. Let $\mathfrak{A} = (\mathfrak{F}, f)$ be given. For each $w \in W$, let $[w] = \{w' \mid \mathfrak{A}, w \sim_\Phi \mathfrak{A}, w'\}$. We first use a partition refinement procedure to compute, in polynomial time, a formula $\psi_{[w]} \in \text{ML}(\Phi)$ such that $\mathfrak{A}, w' \models \psi_{[w]}$ iff $w' \in [w]$.

This procedure works in the same way one would construct a characteristic formula for a σ -bisimulation class of a finite model. That is to say, we start with the labeled partition $\mathcal{I}_0 = \{(W_\psi, \psi) \mid \psi \in \Phi\}$, where $W_\psi = \{w \in W \mid \mathfrak{A}, w \models \psi\}$. Now, consider any $(J, \psi), (J', \psi') \in \mathcal{I}_m$, and let $J^+ = \{j \in J \mid \exists j' \in J' : (j, j') \in R\}$ and $J^- = \{j \in J \mid \forall j' \in J' : (j, j') \notin R\}$. If both J^+ and J^- are non-empty, we obtain $\mathcal{I}_{m+1}$ from $\mathcal{I}_m$ by replacing (J, ψ) with $(J^+, \psi \wedge \Diamond \psi')$ and $(J^-, \psi \wedge \neg \Diamond \psi')$.

Once no further refinements are possible, which happens after at most N steps, the labeled partition will be of the form $\mathcal{I} = \{([w], \psi_{[w]}) \mid w \in W\}$, where $\mathfrak{A}, w' \models \psi_{[w]}$ if and only if $w' \in [w]$. We then define

$$\delta_{\mathfrak{A}} = \psi_{[0]} \wedge \left(\bigwedge_{w \in W} \psi_{[w]} \right) \wedge \left(\bigwedge_{w \in W} \bigwedge_{w' \in W} \bigwedge_{\psi \in \Phi} \psi_{[w]} \rightarrow \pm \Diamond \psi_{[w']} \right)$$

where, $\psi_{[w]} \rightarrow \pm \Diamond \psi_{[w']}$ is equal to $\psi_{[w]} \rightarrow \Diamond \psi_{[w']}$ if there are $j \in [w]$ and $j' \in [w']$ such that $(j, j') \in R$, and equal to $\psi_{[w]} \rightarrow \neg \Diamond \psi_{[w']}$ otherwise. This $\delta_{\mathfrak{A}}$ has the desired properties.

Fix now a tabular quasi-normal modal logic L . Since L is tabular, there is a finite set of rooted finite frames $\mathcal{F}_L = \{(\mathfrak{F}_1, 0), \dots, (\mathfrak{F}_n, 0)\}$ with $L = \text{Log}(\mathcal{F}_L)$. Fix such a set. Note that we assume without loss of generality that the root of each $\mathfrak{F}_i$ is 0. We will often be sloppy and write $\mathfrak{F} \in \mathcal{F}$ instead of $(\mathfrak{F}, 0) \in \mathcal{F}$. Let $N = \max\{|\mathfrak{F}| \mid \mathfrak{F} \in \mathcal{F}_L\}$. By an *abstract Φ -model for L* we will mean an abstract Φ -model whose frame belongs to $\mathcal{F}_L$. If $\mathfrak{A}$ is an abstract Φ -model for L , its *Φ -bisimulation class* with respect to L , denoted $[\mathfrak{A}]_L$, is the set of abstract Φ -models $\mathfrak{A}'$ for L such that $\mathfrak{A}, 0 \sim_\Phi \mathfrak{A}', 0$. A formula $\delta \in \text{ML}(\Phi)$ is an *abstract Φ -class identifier for $[\mathfrak{A}]_L$* if it defines the class $[\mathfrak{A}]_L$ in the sense that for every abstract Φ -model $\mathfrak{A}'$ for L , we have $\mathfrak{A}', 0 \models \delta$ if, and only if, $\mathfrak{A}' \in [\mathfrak{A}]_L$. We call δ an *abstract Φ -class identifier for L* if there is an abstract Φ -model $\mathfrak{A}$ for L such that δ is an abstract Φ -class identifier for $[\mathfrak{A}]_L$. If δ is an abstract Φ -class identifier for $[\mathfrak{A}]_L$ we also write $[\delta]_L$ for $[\mathfrak{A}]_L$.

Note that the formula $\delta_{\mathfrak{A}}$ from Proposition 4.6 is an abstract Φ -class identifier for $[\mathfrak{A}]_L$ since N is fixed as $N = \max\{|\mathfrak{F}| \mid \mathfrak{F} \in \mathcal{F}_L\}$.

► **Example 4.7.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be as in Example 4.4. The formula $\varphi_2 \wedge \Diamond \varphi_1 \wedge \Diamond \varphi_3 \wedge \Box(\varphi_1 \vee \varphi_3)$ is an abstract Φ -class identifier for $[\mathfrak{A}_1]_{L_{\mathfrak{S}}} = [\mathfrak{A}_2]_{L_{\mathfrak{S}}}$.

► **Definition 4.8.** A σ -encoding for L is a pair $(\Gamma, \{\Delta_{\Phi}\}_{\Phi \in \Gamma})$ where Γ is a set of σ -EMEs and Δ_{Φ} is a set of abstract Φ -class identifiers for L such that for every model $\mathfrak{M}$ based on a frame in $\mathcal{F}_L$, there are a $\Phi \in \Gamma$ and $\delta \in \Delta_{\Phi}$ such that Φ is a σ -cover of $\mathfrak{M}$ and $\mathfrak{M}, 0 \models \delta$.

The following proposition summarizes the relevant results proved above in a more compact way:

► **Proposition 4.9.** Fix a tabular quasi-normal modal logic L . Given a signature σ , we can compute in polynomial time a σ -encoding for L .

Proof. We first use Lemma 4.2 to construct, for each $\mathfrak{F} \in \mathcal{F}_L$, in time polynomial in $|\sigma|$, a set $\Gamma_{\mathfrak{F}}$ of σ -EMEs such that every model based on $\mathfrak{F}$ has a σ -cover in $\Gamma_{\mathfrak{F}}$. Let $\Gamma = \bigcup_{\mathfrak{F} \in \mathcal{F}_L} \Gamma_{\mathfrak{F}}$. Next, we apply Proposition 4.6 to obtain, for each $\Phi \in \Gamma$ and for each abstract Φ -model $\mathfrak{A}$ for L , an abstract Φ -class identifier $\delta_{\mathfrak{A}}$ for L (recall that we fixed $N = \max\{|\mathfrak{F}| \mid \mathfrak{F} \in \mathcal{F}_L\}$). Note that there are at most $\Sigma_{\mathfrak{F} \in \mathcal{F}_L} |\Phi|^{|\mathfrak{F}|}$ many abstract Φ -models for L , which is polynomial in $|\Phi|$ and hence in $|\sigma|$. Let Δ be the set of all these abstract class identifiers. It follows from the construction that (Γ, Δ) is a σ -encoding for L . ◀

4.2 From Modal Logic to Propositional Logic and Back

In this section, we use propositional formulas to speak about pointed models. Given $\mathfrak{F} = (W, R)$ with root 0 and a set σ of atoms, we form the set of atoms $\sigma_{\mathfrak{F}} = \sigma \times W$. For notational convenience we denote $(p, w) \in \sigma_{\mathfrak{F}}$ as p_w . Then $\mathfrak{M} = (\mathfrak{F}, V)$ defines a propositional valuation $v_{\mathfrak{M}} : \sigma \times W \rightarrow \{0, 1\}$ for the language $\text{PL}(\sigma_{\mathfrak{F}})$ by setting

$$v_{\mathfrak{M}}(p_w) = \begin{cases} 1 & \text{if } \mathfrak{M}, w \models p \\ 0 & \text{if } \mathfrak{M}, w \models \neg p \end{cases}$$

Observe that, conversely, every propositional valuation v for $\text{PL}(\sigma_{\mathfrak{F}})$ defines a modal σ -model $\mathfrak{M}_v = (\mathfrak{F}, V_v)$ by setting $\mathfrak{M}_v, w \models p$ iff $v(p_w) = 1$. For every modal formula in $\text{ML}(\sigma)$ we can easily find an ‘equivalent’ propositional formula in $\text{PL}(\sigma_{\mathfrak{F}})$ which we define next.

► **Definition 4.10.** The translation functions $tr_{\mathfrak{F}, w} : \text{ML}(\sigma) \rightarrow \text{PL}(\sigma_{\mathfrak{F}})$ with $w \in W$ are defined by mutual induction:

$$\begin{aligned} tr_{\mathfrak{F}, w}(p) &= p_w \\ tr_{\mathfrak{F}, w}(\neg \varphi) &= \neg tr_{\mathfrak{F}, w}(\varphi) \\ tr_{\mathfrak{F}, w}(\varphi \wedge \psi) &= tr_{\mathfrak{F}, w}(\varphi) \wedge tr_{\mathfrak{F}, w}(\psi) \\ tr_{\mathfrak{F}, w}(\Diamond \varphi) &= \bigvee_{(w, w') \in R} tr_{\mathfrak{F}, w'}(\varphi) \end{aligned}$$

We summarize the main properties of $tr_{\mathfrak{F}, w}(\varphi)$.

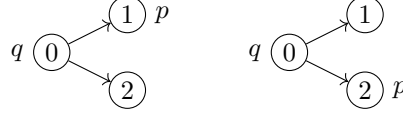
► **Proposition 4.11.** Let $\mathfrak{F} = (W, R)$ be a finite frame with root 0, $w \in W$, and $\mathfrak{M}$ a model based on $\mathfrak{F}$.

1. for every $\varphi \in \text{ML}(\sigma)$, we have $\mathfrak{M}, w \models \varphi$ iff $v_{\mathfrak{M}} \models tr_{\mathfrak{F}, w}(\varphi)$.
2. for every $\varphi \in \text{ML}(\sigma)$, $(\mathfrak{F}, 0) \models \varphi$ iff $tr_{\mathfrak{F}, 0}(\varphi)$ is a tautology.

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Note that not every propositional formula over $\sigma \times W$ is equivalent to the translation of a modal formula.

► **Example 4.12.** Consider the following pointed models $(\mathfrak{M}_3, 0)$ and $(\mathfrak{M}_4, 0)$:



Since these models are σ -bisimilar for $\sigma = \{p, q\}$, no formula in $ML(\sigma)$ can distinguish between them. It follows that no translation $tr_{\mathfrak{F},0}(\varphi)$ of a formula φ in $ML(\sigma)$ can distinguish between the corresponding propositional models $v_{\mathfrak{M}_3}$ and $v_{\mathfrak{M}_4}$. This implies that no translation $tr_{\mathfrak{F},0}(\varphi)$ of a formula φ in $ML(\sigma)$ is equivalent to the propositional formula p_1 , since that formula does distinguish between the two models.

This concludes the presentation of the “forward” translation tr , i.e., from modal formulas to propositional formulas. We also need a reverse translation rt , in order to translate the propositional uniform interpolant back to a modal formula. For this, we make use of the techniques we developed in the previous subsection. In particular, we make use of Proposition 4.9 and Lemma 4.1.

Fix a tabular quasi-normal modal logic L , and let $\mathcal{F}_L$ be a finite set of finite rooted frames such that $L = \text{Log}(\mathcal{F}_L)$. Let $N = \max\{|\mathfrak{F}'| \mid \mathfrak{F}' \in \mathcal{F}_L\}$. Recall that we denote the root of each $\mathfrak{F} \in \mathcal{F}_L$ by 0.

► **Definition 4.13.** Fix a tabular quasi-normal modal logic L , and let $\mathfrak{F} \in \mathcal{F}_L$. Then $rt_{\mathfrak{F},L} : PL(\sigma_{\mathfrak{F}}) \rightarrow ML(\sigma)$ is given by

$$rt_{\mathfrak{F},L}(\xi) = \bigwedge_{\Phi \in \Gamma} \bigwedge_{\delta \in \Delta_{\Phi}} \left((cover(\sigma, \Phi) \wedge \delta) \rightarrow \bigvee_{(\mathfrak{F}, f) \in [\delta]_L} \xi^f \right)$$

where $(\Gamma, \{\Delta_{\Phi}\}_{\Phi \in \Gamma})$ is a σ -encoding for L as given by Proposition 4.9, and ξ^f is the result of replacing, in ξ , each indexed atom $p_w \in \sigma_{\mathfrak{F}}$ by $\Diamond^{\leq N}(f(w) \wedge p)$.

By the results in the previous subsection, all computations involved in this translation can be performed in polynomial time (assuming L is fixed, and hence $N = \max_{\mathfrak{F} \in \mathcal{F}_L} |\mathfrak{F}|$ can be treated as a constant). In particular, note that the number of $(\mathfrak{F}, f) \in [\delta]_L$ is at most $|\Phi|^{|\delta|} \leq |\Phi|^N$, hence is also polynomial.

► **Proposition 4.14.** Fix a tabular quasi-normal modal logic L . Let $\mathfrak{M}, 0$ be any pointed model based on a pointed frame in $\mathcal{F}_L$. Furthermore, let $\mathfrak{F} \in \mathcal{F}_L$ and let $\xi \in PL(\sigma_{\mathfrak{F}})$. Then the following are equivalent:

1. $\mathfrak{M}, 0 \models rt_{\mathfrak{F},L}(\xi)$,
2. there is a model $\mathfrak{M}'$ based on the frame $\mathfrak{F}$ such that $\mathfrak{M}, 0 \sim_{\sigma} \mathfrak{M}', 0$ and $v_{\mathfrak{M}'} \models \xi$.

The proof uses the following lemma, which we prove first.

► **Lemma 4.15.** Let Φ be a σ -cover of $\mathfrak{M}$, Z a σ -bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ such that wZw' and $\varphi \in \Phi$ the formula such that $\mathfrak{M}', w' \models \varphi$. Then for every $p \in \sigma$, $v_{\mathfrak{M}'} \models p_{w'}$ if, and only if, $\mathfrak{M}, 0 \models \Diamond^{\leq N}(\varphi \wedge p)$.

Proof. ($\Rightarrow$) Suppose $v_{\mathfrak{M}'} \models p_{w'}$. Then $\mathfrak{M}', w' \models p$. By assumption, $\mathfrak{M}', w' \models \varphi$, so $\mathfrak{M}', w' \models \varphi \wedge p$. As Z is a σ -bisimulation and $\varphi \wedge p \in \text{ML}(\sigma)$, this implies that $\mathfrak{M}, w \models \varphi \wedge p$, and therefore $\mathfrak{M}, 0 \models \Diamond^{\leq N}(\varphi \wedge p)$.

($\Leftarrow$) Suppose $\mathfrak{M}, 0 \models \Diamond^{\leq N}(\varphi \wedge p)$. Because Z is a σ -bisimulation and $\mathfrak{M}', w' \models \varphi$, we have $\mathfrak{M}, w \models \varphi$. As Φ is a σ -cover of $\mathfrak{M}$, the value of p is constant across all worlds that satisfy φ . From $\mathfrak{M}, 0 \models \Diamond^{\leq N}(\varphi \wedge p)$ it therefore follows that $\mathfrak{M}, w \models p$, which implies that $\mathfrak{M}', w' \models p$. It follows that $v_{\mathfrak{M}'} \models p_{w'}$. $\blacktriangleleft$

Proof of Proposition 4.14. ($\Rightarrow$) Suppose $\mathfrak{M}, 0 \models rt_{\mathfrak{F}}(\xi)$. Because $(\Gamma, \{\Delta_{\Phi}\}_{\Phi \in \Gamma})$ is a σ -encoding, there are $\Phi \in \Gamma, \delta \in \Delta_{\Phi}$ such that Φ is a σ -cover of $\mathfrak{M}$ and $\mathfrak{M}, 0 \models \delta$. Let $\mathfrak{A}$ be the Φ -abstraction of $\mathfrak{M}$. We then have $\mathfrak{M}, 0 \models \text{cover}(\sigma, \Phi) \wedge \delta$, so the antecedent of $rt_{\mathfrak{F}}(\xi)$ is satisfied for these Φ and δ . By assumption, $\mathfrak{M}, 0 \models rt_{\mathfrak{F}}(\xi)$, so the consequent of $rt_{\mathfrak{F}}(\xi)$ must also hold, i.e., $\mathfrak{M}, 0 \models \bigvee_{(\mathfrak{F}, f) \in [\delta]_L} \xi^f$. Let $\mathfrak{A}' = (\mathfrak{F}, f) \in [\delta]_L$ be such that $\mathfrak{M}, 0 \models \xi^f$.

We are given that $\mathfrak{A}' \in [\delta]_L$. Because $\mathfrak{M}, 0 \models \text{cover}(\sigma, \Phi) \wedge \delta$ we also have $\mathfrak{A} \in [\delta]_L$. So there is an abstract Φ -bisimulation Z between $\mathfrak{A}$ and $\mathfrak{A}'$. It follows that there is some $\mathfrak{M}' = (\mathfrak{F}', V')$ such that $\mathfrak{A}'$ is the Φ -abstraction of $\mathfrak{M}'$ and Z is a σ -bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Because $\mathfrak{A}' = (\mathfrak{F}, f)$ is the Φ -abstraction of $\mathfrak{M}'$, we have $\mathfrak{M}', w' \models f(w')$ for all $w' \in W'$. The conditions of Lemma 4.15 are therefore satisfied, which implies that $\mathfrak{M}, 0 \models \Diamond^{\leq N}(f(w') \wedge p)$ if, and only if, $v_{\mathfrak{M}'} \models p_{w'}$. From $\mathfrak{M}, 0 \models \xi^f$ it therefore follows that $v_{\mathfrak{M}'} \models \xi$. We have seen already that $\mathfrak{M}, 0$ and $\mathfrak{M}', 0$ are σ -bisimilar, so this completes the left-to-right direction of the proof.

($\Leftarrow$) Suppose that $\mathfrak{M}' = (\mathfrak{F}, V')$ is σ -bisimilar to $\mathfrak{M}$ and that $v_{\mathfrak{M}'} \models \xi$. We must show that $\mathfrak{M}, 0 \models rt_{\mathfrak{F}}(\xi)$. Let $\Phi \in \Gamma$ and $\delta \in \Delta_{\Phi}$ be such that the antecedent of $rt_{\mathfrak{F}}(\xi)$ is satisfied, i.e., $\mathfrak{M}, 0 \models \text{cover}(\sigma, \Phi) \wedge \delta$. From $\mathfrak{M}, 0 \models \text{cover}(\sigma, \Phi)$ it follows that Φ is a σ -cover of $\mathfrak{M}$. Let $\mathfrak{A}$ be the Φ -abstraction of $\mathfrak{M}$. Because $\mathfrak{M}, 0 \models \delta$ we also have $\mathfrak{A}, 0 \models \delta$, so $\mathfrak{A} \in [\delta]_L$.

Since $\mathfrak{M}, 0 \sim_{\Phi} \mathfrak{M}', 0$, we have that Φ is also a σ -cover of $\mathfrak{M}'$, and that $\mathfrak{M}', 0 \models \delta$. Let $\mathfrak{A}' = (\mathfrak{F}', f')$ be the Φ -abstraction of $\mathfrak{M}'$. Then $\mathfrak{A}', 0 \models \delta$, and therefore $\mathfrak{A}' \in [\delta]_L$. Furthermore, for every $w' \in W'$ we have $\mathfrak{M}', w' \models f(w')$. By Lemma 4.15 we now have $v_{\mathfrak{M}'} \models p_{w'}$ if, and only if, $\mathfrak{M}, 0 \models \Diamond^{\leq N}(f(w') \wedge p)$. Since, by assumption, $v_{\mathfrak{M}'} \models \xi$, this implies that $\mathfrak{M}, 0 \models \xi^f$. The consequent of $rt_{\mathfrak{F}}(\xi)$ is therefore also satisfied in $\mathfrak{M}, 0$. As this holds for every Φ and δ for which the antecedent is satisfied, we have $\mathfrak{M}, 0 \models rt_{\mathfrak{F}}(\xi)$, which was to be shown. $\blacktriangleleft$

► Example 4.16. The key observation in the proof of Proposition 4.14 is that if $\mathfrak{M}, 0$ and $\mathfrak{M}', 0$ are σ -bisimilar, with abstractions $\mathfrak{A} = (\mathfrak{F}, f)$ and $\mathfrak{A}' = (\mathfrak{F}', f')$, respectively, then $v_{\mathfrak{M}'} \models \xi$ if and only if $\mathfrak{M}, 0 \models \xi^{f'}$. This ensures that if ξ is true in some model σ -bisimilar to $\mathfrak{M}, 0$, then at least one of the disjuncts of $\bigvee_{(\mathfrak{F}, f) \in [\delta]_L} \xi^f$ is true in $\mathfrak{M}, 0$.

To illustrate this, let $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}_1, \mathfrak{A}_2$ and Z be as in Example 4.4. Furthermore, let f_2 be the function such that $\mathfrak{A}_2 = (\mathfrak{G}, f_2)$ and let $\mathfrak{M}'_2$ be the model such that Z is a σ -bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}'_2$. Note that $\mathfrak{A}_2$ is the Φ -abstraction of both $\mathfrak{M}_2$ and $\mathfrak{M}'_2$.

Take $\xi = s_1 \in \text{PL}(\sigma_{\mathfrak{G}})$. Then ξ is not true in $v_{\mathfrak{M}_1}$ or $v_{\mathfrak{M}_2}$, but true in $v_{\mathfrak{M}'_2}$. So ξ is true in some model σ -bisimilar to $\mathfrak{M}_1$, namely $\mathfrak{M}'_2$.

Because f_2 is the function associated with the abstraction of $\mathfrak{M}'_2$, we then have $\mathfrak{M}_1 \models \xi^{f_2}$. Indeed, $\xi^{f_2} = \Diamond^{\leq N}(s \wedge f_2(w_1)) = \Diamond^{\leq N}(s \wedge \neg p \wedge \neg r)$, which holds in $\mathfrak{M}_1, 0$.

The following result implies Theorem 1.1 for $L = \text{Log}(\mathcal{F})$ for a singleton $\mathcal{F} = \{(\mathfrak{F}, 0)\}$.

► Theorem 4.17. Fix a tabular quasi-normal modal logic L with $L = \text{Log}(\mathcal{F})$ for a singleton $\mathcal{F} = \{(\mathfrak{F}, 0)\}$. Let $\varphi \in \text{ML}$ and let $\sigma \subseteq \text{sig}(\varphi)$. If $\xi \in \text{PL}(\sigma_{\mathfrak{F}})$ is a propositional uniform $\sigma_{\mathfrak{F}}$ -interpolant for $\text{tr}_{\mathfrak{F}, 0}(\varphi)$, then $rt_{\mathfrak{F}, L}(\xi)$ is a strongest $L(\sigma)$ -implicate of φ .

Proof. (1) We show that $\varphi \rightarrow rt_{\mathfrak{F},L}(\xi) \in L$. Assume $\mathfrak{M}, 0 \models \varphi$ with $\mathfrak{M}$ based on $\mathfrak{F}$. Then $v_{\mathfrak{M}} \models tr_{\mathfrak{F},0}(\varphi)$. Hence $v_{\mathfrak{M}} \models \xi$. By the backward direction of Proposition 4.14 (using the identity bisimulation), $\mathfrak{M}, 0 \models rt_{\mathfrak{F},L}(\xi)$.

(2) Suppose that $\varphi \rightarrow \chi \in L$ for some $\chi \in ML(\sigma)$. We must show that $rt_{\mathfrak{F},L}(\xi) \rightarrow \chi \in L$. Assume $\mathfrak{M}, 0 \models rt_{\mathfrak{F},L}(\xi)$. By Proposition 4.14, there is a σ -bisimilar $\mathfrak{M}', 0$ based on $\mathfrak{F}$ with $v_{\mathfrak{M}'} \models \xi$. Next note that by Proposition 4.11, $tr_{\mathfrak{F},0}(\varphi) \rightarrow tr_{\mathfrak{F},0}(\chi)$ is a tautology. Hence, as ξ is a uniform $\sigma_{\mathfrak{F}}$ -interpolant for $tr_{\mathfrak{F},0}(\varphi)$, $\xi \rightarrow tr_{\mathfrak{F},0}(\chi)$ is a tautology. It follows that $v_{\mathfrak{M}'} \models tr_{\mathfrak{F},0}(\chi)$ and so $\mathfrak{M}', 0 \models \chi$. Therefore, by invariance under σ -bisimulations, $\mathfrak{M}, 0 \models \chi$. $\blacktriangleleft$

4.3 From Single Frames to Multiple Frames

We now lift Theorem 4.17 to tabular quasi-normal logics defined by a *finite set* of finite rooted frames. Let as before $\mathcal{F}_L$ be a finite set of finite pointed frames and $L = \text{Log}(\mathcal{F}_L)$. We first show that the strongest $L(\sigma)$ -implicate of a modal formula φ can be obtained in polynomial time from the propositional uniform $\sigma_{\mathfrak{F}}$ -interpolants of $tr_{\mathfrak{F}}(\varphi)$, $\mathfrak{F} \in \mathcal{F}_L$. To strengthen this result to a reduction of L -implicate computation to a *single* propositional uniform interpolant, we then generalize the translation $tr_{\mathfrak{F}}$ to sets $\mathcal{F}_L$ of frames and show how uniform interpolants for $tr_{\mathcal{F}}(\varphi)$ are obtained from uniform interpolants for $tr_{\mathfrak{F}}(\varphi)$, $\mathfrak{F} \in \mathcal{F}_L$.

► **Theorem 4.18.** *Fix a tabular quasi-normal modal logic L with $L = \text{Log}(\mathcal{F}_L)$. Let $\varphi \in ML$ and $\sigma \subseteq \text{sig}(\varphi)$. Let $\xi_{\mathfrak{F}} \in PL(\sigma_{\mathfrak{F}})$ be propositional uniform $\sigma_{\mathfrak{F}}$ -interpolants for $tr_{\mathfrak{F}}(\varphi)$, for $\mathfrak{F} \in \mathcal{F}_L$. Then $\bigvee_{\mathfrak{F} \in \mathcal{F}_L} rt_{\mathfrak{F},L}(\xi_{\mathfrak{F}})$ (cf. Definition 4.13) is a strongest $L(\sigma)$ -implicate of φ .*

Proof. (1) We show that $\varphi \rightarrow \bigvee_{\mathfrak{F} \in \mathcal{F}_L} rt_{\mathfrak{F},L}(\xi_{\mathfrak{F}}) \in L$. Assume $\mathfrak{M}, 0 \models \varphi$ with $\mathfrak{M}$ based on $\mathfrak{F} \in \mathcal{F}_L$. Then $v_{\mathfrak{M}} \models tr_{\mathfrak{F}}(\varphi)$. Hence $v_{\mathfrak{M}} \models \xi_{\mathfrak{F}}$. By Proposition 4.14, $\mathfrak{M}, 0 \models rt_{\mathfrak{F},L}(\xi_{\mathfrak{F}})$, as required.

(2) We show that $\bigvee_{\mathfrak{F} \in \mathcal{F}_L} rt_{\mathfrak{F},L}(\xi_{\mathfrak{F}}) \rightarrow \chi \in L$ for all $ML(\sigma)$ -formulas χ with $\varphi \rightarrow \chi \in L$. To this end, it suffices to show that if $\mathfrak{M}, 0 \models \bigvee_{\mathfrak{F} \in \mathcal{F}_L} rt_{\mathfrak{F},L}(\xi_{\mathfrak{F}})$ for some $\mathfrak{M}$ based on $\mathfrak{F} \in \mathcal{F}_L$, then there exists a σ -bisimilar $\mathfrak{M}'$ with $\mathfrak{M}', 0 \models \varphi$ based on some $\mathfrak{F}' \in \mathcal{F}_L$. So assume $\mathfrak{M}, 0 \models \bigvee_{\mathfrak{F} \in \mathcal{F}_L} rt_{\mathfrak{F},L}(\xi_{\mathfrak{F}})$ with $\mathfrak{M}$ based on $\mathfrak{F}$. Then we can take $\mathfrak{F}' \in \mathcal{F}_L$ with $\mathfrak{M}, 0 \models rt_{\mathfrak{F}',L}(\xi_{\mathfrak{F}'})$. By Proposition 4.14, there is a σ -bisimilar $\mathfrak{M}'$ based on $\mathfrak{F}'$ with $v_{\mathfrak{M}'} \models \xi_{\mathfrak{F}'}$. As $\xi_{\mathfrak{F}'}$ is a uniform $\sigma_{\mathfrak{F}'}$ -interpolant for $tr_{\mathfrak{F}'}(\varphi)$, we can modify the values of non- $\sigma_{\mathfrak{F}'}$ atoms in $v_{\mathfrak{M}'}$ to obtain a model v with $v \models tr_{\mathfrak{F}'}(\varphi)$. But then $\mathfrak{M}'', 0 \sim_{\sigma} \mathfrak{M}', 0 \sim_{\sigma} \mathfrak{M}, 0$ and $\mathfrak{M}'', 0 \models \varphi$ for the model $\mathfrak{M}''$ based on $\mathfrak{F}'$ with $v_{\mathfrak{M}''} = v$. $\blacktriangleleft$

The above theorem can be interpreted as a Turing reduction: it provides an efficient algorithm for computing a strongest $L(\sigma)$ -implicate for a given modal formula, where the algorithm is allowed to ask one or more queries to an oracle that computes propositional uniform interpolants. In particular, it follows that if propositional uniform interpolants can be computed in polynomial time, then strongest $L(\sigma)$ -implicates can be computed in polynomial time. Theorem 1.1 from the introduction (restated below) is slightly stronger, as it provides an algorithm that asks only one oracle query. Our next aim is therefore to strengthen Theorem 4.18 to bridge this gap.

► **Theorem 1.1.** *Let L be a tabular quasi-normal modal logic. We can associate to each signature σ of propositional variables a companion signature $\hat{\sigma}$ such that there are poly-time translations*

$$tr_L : ML(\sigma) \rightarrow PL(\hat{\sigma}) \quad \text{and} \quad rt_L : PL(\hat{\sigma}) \rightarrow ML(\sigma)$$

such that for every $\varphi \in ML(\sigma)$, signature $\tau \subseteq \text{sig}(\varphi)$, and every uniform $\hat{\tau}$ -interpolant ψ of $tr_L(\varphi)$ in propositional logic, $rt_L(\psi)$ is a strongest $L(\tau)$ -implicate for φ .

Proof. We omit some details of the proof, a sketch of the proof strategy is as follows. First we extend translation $tr_{\mathfrak{F}}$ to a translation tr_L for any tabular quasi-normal logic L . Fix $L = \text{Log}(\mathcal{F}_L)$ for a finite set $\mathcal{F}_L$ of finite rooted frames. We assume that for $(\mathfrak{F}, w), (\mathfrak{F}', w') \in \mathcal{F}_L$ with $(\mathfrak{F}, w) \neq (\mathfrak{F}', w')$ the sets of worlds in $\mathfrak{F}$ and $\mathfrak{F}'$ are mutually disjoint (and so do not use 0 to denote the roots of frames in $\mathcal{F}_L$). Take atoms $r_{\mathfrak{F}, w}$, for $(\mathfrak{F}, w) \in \mathcal{F}_L$. They are used to identify the frame $(\mathfrak{F}, w) \in \mathcal{F}_L$ in which we evaluate a modal formula. For any signature σ , let $\sigma_L = \bigcup_{(\mathfrak{F}, w) \in \mathcal{F}_L} \sigma_{\mathfrak{F}, w}$. The signatures $\sigma_{\mathfrak{F}, w}$ are assumed to be mutually disjoint and also disjoint from $\{r_{\mathfrak{F}, w} \mid (\mathfrak{F}, w) \in \mathcal{F}_L\}$. Let $\hat{\sigma} = \sigma_L \cup \{r_{\mathfrak{F}, w} \mid (\mathfrak{F}, w) \in \mathcal{F}_L\}$. Next let for any modal formula φ ,

$$tr_L(\varphi) = \text{Unique}(\mathcal{F}_L) \wedge \bigwedge_{(\mathfrak{F}, w) \in \mathcal{F}_L} (r_{\mathfrak{F}, w} \rightarrow tr_{\mathfrak{F}, w}(\varphi)).$$

where

$$\text{Unique}(\mathcal{F}_L) = \bigvee_{(\mathfrak{F}, w) \in \mathcal{F}_L} (r_{\mathfrak{F}, w} \wedge \bigwedge_{(\mathfrak{G}, v) \in \mathcal{F}_L \setminus \{(\mathfrak{F}, w)\}} \neg r_{\mathfrak{G}, v})$$

Hence $tr_L(\varphi) \in \text{PL}(\hat{\sigma})$ uses $\text{Unique}(\mathcal{F}_L)$ to pick $(\mathfrak{F}, w) \in \mathcal{F}_L$ and then states that $tr_{\mathfrak{F}, w}(\varphi)$ holds.

We next define the translation rt_L (which is meaningful only if applied to propositional uniform interpolants). To this end, we first construct from a propositional uniform $\hat{\sigma}$ -interpolant of $tr_L(\varphi)$ propositional uniform $\sigma_{\mathfrak{F}, w}$ -interpolants for $tr_{\mathfrak{F}, w}(\varphi)$, for every $(\mathfrak{F}, w) \in \mathcal{F}_L$. Assume φ is given and $\sigma \subseteq \text{sig}(\varphi)$. Let ξ be any propositional uniform $\hat{\sigma}$ -interpolant of $tr_L(\varphi)$. Then obtain for $(\mathfrak{F}, w) \in \mathcal{F}_L$ the formula $\xi^{\uparrow \mathfrak{F}, w}$ from ξ by

- replacing $r_{\mathfrak{F}, w}$ by $\top$,
- replacing all $r_{\mathfrak{G}, v}$ with $(\mathfrak{G}, v) \in \mathcal{F}_L \setminus \{(\mathfrak{F}, w)\}$ by $\perp$, and
- replacing all $p_{\mathfrak{G}, v}$ with v not in $(\mathfrak{F}, w)$ by $\perp$.

It is easy to see that $\xi^{\uparrow \mathfrak{F}, w}$ is a propositional uniform $\sigma_{\mathfrak{F}, w}$ -interpolant for $tr_{\mathfrak{F}, w}(\varphi)$, for every $(\mathfrak{F}, w) \in \mathcal{F}_L$. Now we simply define for any $\xi \in \text{PL}(\hat{\sigma})$,

$$rt_L(\xi) = \bigvee_{(\mathfrak{F}, w) \in \mathcal{F}_L} rt_{(\mathfrak{F}, w), L}(\xi^{\uparrow \mathfrak{F}, w})$$

and show using Theorem 4.18 that rt_L is as required. $\blacktriangleleft$

5 Constructing Craig Interpolants via Propositional Logic

We show that if a tabular quasi-normal logic $L = \text{Log}(\mathcal{F}_L)$ has CIP, then one can compute Craig interpolants for an implication $\varphi \rightarrow \psi \in L$ from propositional Craig interpolants for $tr_L(\varphi) \rightarrow tr_L(\psi)$ in polynomial time. To this end, we first give a criterion for CIP for tabular quasi-normal logics which shows that L has CIP iff no non-trivial bisimulations between models based on frames in $\mathcal{F}_L$ exist. We then show that the backward translation $\bigvee_{\mathfrak{F} \in \mathcal{F}_L} rt_{\mathfrak{F}, L}(\xi_{\mathfrak{F}})$ is a Craig interpolant for $\varphi \rightarrow \psi$ if the $\xi_{\mathfrak{F}}$ are Craig interpolants for $tr_{\mathfrak{F}}(\varphi) \rightarrow tr_{\mathfrak{F}}(\psi)$ in $\text{Log}(\mathfrak{F})$ for $\mathfrak{F} \in \mathcal{F}_L$.

We close with a discussion of Craig interpolants for modal logics that do not have CIP.

Let $L = \text{Log}(\mathcal{F}_L)$ with $\mathcal{F}_L = \{(\mathfrak{F}_1, 0), \dots, (\mathfrak{F}_n, 0)\}$. Clearly, we may assume that $\mathcal{F}_L$ is *reduced* in the sense that $\text{Log}(\mathfrak{F}_i, 0) \not\subseteq \text{Log}(\mathfrak{F}_j, 0)$ for $i \neq j$. We start by giving a criterion for CIP of L . The σ -*reduct* of a model $\mathfrak{M}$ is the restriction of $\mathfrak{M}$ to atoms in σ . An *isomorphism* between the σ -reducts of pointed models $\mathfrak{M}_1, w_1$ and $\mathfrak{M}_2, w_2$ is an isomorphism f between the underlying frames with $f(w_1) = w_2$ such that $\mathfrak{M}_1, w \models p$ iff $\mathfrak{M}_2, f(w) \models p$, for all $p \in \sigma$.

► **Theorem 5.1.** *Let $L = \text{Log}(\mathcal{F}_L)$ be a tabular quasi-normal logic with $\mathcal{F}_L$ a reduced set of finite rooted frames. Then L has CIP if, and only if, for all signatures σ the following holds: if $\mathfrak{M}_1, 0 \sim_\sigma \mathfrak{M}_2, 0$ with $\mathfrak{M}_1$ a model based on $\mathfrak{F}_i \in \mathcal{F}_L$ and $\mathfrak{M}_2$ a model based on $\mathfrak{F}_j \in \mathcal{F}_L$, then the σ -reducts of $\mathfrak{M}_1, 0$ and $\mathfrak{M}_2, 0$ are isomorphic and, in particular, $i = j$.*

Proof. We use the following criterion for CIP for tabular modal logics which follows immediately from known sufficient conditions [25]:

Fact 1. L has CIP iff for all modal formulas φ, ψ and $\sigma = \text{sig}(\varphi) \cap \text{sig}(\psi)$ the following holds: if there are models $\mathfrak{M}_1, 0 \models \varphi$ and $\mathfrak{M}_2, 0 \models \psi$ with $\mathfrak{M}_1, 0 \sim_\sigma \mathfrak{M}_2, 0$ such that $\mathfrak{M}_1$ is based on $\mathfrak{F}_i \in \mathcal{F}_L$ and $\mathfrak{M}_2$ is based on $\mathfrak{F}_j \in \mathcal{F}_L$, then there is a model $\mathfrak{M}', 0$ based on some $\mathfrak{F}_k \in \mathcal{F}_L$ with $\mathfrak{M}', 0 \models \varphi \wedge \psi$.

The direction from right to left is a direct consequence of Fact 1. We now show the converse direction. Assume $(\mathfrak{F}, 0)$ is a finite rooted frame and $N = \max_{\mathfrak{F} \in \mathcal{F}_L} |\mathfrak{F}|$. Define the *diagram* $\text{diag}(\mathfrak{F}, 0)$ of $(\mathfrak{F}, 0)$ (also known as the splitting-, Jankov-, or canonical formula of $(\mathfrak{F}, 0)$ [35]) as the conjunction of the following formulas (with q_w fresh atoms for w in $\mathfrak{F}$):

$$\begin{aligned} & q_0 \wedge \Box^{\leq N} \bigvee_{v \in W} q_v \\ & \Box^{\leq N} (q_v \rightarrow \neg q_{v'}) \text{ if } v \neq v' \\ & \Box^{\leq N} (q_v \rightarrow \Diamond q_{v'}) \text{ if } v R v' \\ & \Box^{\leq N} (q_v \rightarrow \neg \Diamond q_{v'}) \text{ if not } v R v'. \end{aligned}$$

The following facts about diagrams are straightforward:

Fact 2. (1) If there exists a model $\mathfrak{M}$ based on $\mathfrak{F}_j \in \mathcal{F}_L$ with $\mathfrak{M}, 0 \models \text{diag}(\mathfrak{F}_i, 0)$, then $\text{Log}(\mathfrak{F}_i, 0) \supseteq \text{Log}(\mathfrak{F}_j, 0)$ and so $i = j$.

(2) If $\mathfrak{M}, 0 \models \text{diag}(\mathfrak{F}_i, 0)$ for a model $\mathfrak{M}$ based on $\mathfrak{F}_i \in \mathcal{F}_L$, then for every w in $\mathfrak{F}_i$ there is exactly one $f(w)$ in $\mathfrak{F}_i$ with $\mathfrak{M}, f(w) \models q_w$. f is an automorphism on $(\mathfrak{F}_i, 0)$.

Now assume that L has CIP. Assume a signature σ and $\mathfrak{M}_1, 0 \sim_\sigma \mathfrak{M}_2, 0$ with $\mathfrak{M}_1$ a model based on $\mathfrak{F}_i \in \mathcal{F}_L$ and $\mathfrak{M}_2$ a model based on $\mathfrak{F}_j \in \mathcal{F}_L$ are given. We take disjoint sets of atoms $q_{w,1}$ for w in $\mathfrak{F}_i$ and $q_{w,2}$ for w in $\mathfrak{F}_j$. Denote by diag_1 the diagram of $(\mathfrak{F}_i, 0)$ using the atoms $q_{w,1}$ and by diag_2 the diagram of $(\mathfrak{F}_j, 0)$ using the atoms $q_{w,2}$, respectively. Note that $i = j$ is allowed. We also assume that the atoms of these diagrams are disjoint from σ . We now define descriptions $\delta_\sigma(\mathfrak{M}_k, 0)$ of the σ -reducts of $\mathfrak{M}_k$ by setting for $L_\sigma = \{p, \neg p \mid p \in \sigma\}$ and $k = 1, 2$:

$$\delta_\sigma(\mathfrak{M}_k, 0) = \text{diag}_k \wedge \Box^{\leq N} \left(\bigwedge_{\substack{\mathfrak{M}_k, w \models l \\ l \in L_\sigma}} (q_{w,k} \rightarrow l) \right).$$

It should be clear that $\mathfrak{M}_k, 0 \models \delta_\sigma(\mathfrak{M}_k, 0)$ for $k = 1, 2$. Since L has CIP we can apply Fact 1 because $\sigma = \text{sig}(\delta_\sigma(\mathfrak{M}_1, 0)) \cap \text{sig}(\delta_\sigma(\mathfrak{M}_2, 0))$. Hence, there is a model $\mathfrak{M}'$ based on a $\mathfrak{F}_k \in \mathcal{F}_L$ such that $\mathfrak{M}' \models \delta_\sigma(\mathfrak{M}_1, 0) \wedge \delta_\sigma(\mathfrak{M}_2, 0)$. By Fact 2 (1), $i = j = k$.

By Fact 2 (2), we have automorphisms $f_1 : (\mathfrak{F}_i, 0) \rightarrow (\mathfrak{F}_i, 0)$ and $f_2 : (\mathfrak{F}_j, 0) \rightarrow (\mathfrak{F}_j, 0)$ with $\mathfrak{M}', f_1(w) \models q_{w,1}$ and $\mathfrak{M}', f_2(w) \models q_{w,2}$ for all w in $\mathfrak{F}_i$. f_1 and f_2 are invariant for σ -atoms in the sense that for all w in $\mathfrak{F}_i$ and $p \in \sigma$: $\mathfrak{M}_k, w \models p$ iff $\mathfrak{M}', f_k(w) \models p$, for $k = 1, 2$. It follows that $f_1 \circ f_2^{-1}$ is an isomorphism between the σ -reducts of $(\mathfrak{M}_1, 0)$ and $(\mathfrak{M}_2, 0)$, as required. ◀

The following example illustrates Theorem 5.1.

► **Example 5.2.** (1) Let $\mathfrak{G}$ and $L_{\mathfrak{G}}$ be from Example 3.2. Observe that for $\sigma = \{p\}$ and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on $\mathfrak{G}$ with p true in 1 in $\mathfrak{M}_1$ and p true in 1, 2 in $\mathfrak{M}_2$ we have $\mathfrak{M}_1, 0 \sim_\sigma \mathfrak{M}_2, 0$ but $\mathfrak{M}_1, 0$ and $\mathfrak{M}_2, 0$ are not isomorphic. Hence $L_{\mathfrak{G}}$ does not have CIP.

(2) Observe that any two models $\mathfrak{M}_1, 0$ and $\mathfrak{M}_2, 0$ based on reflexive frames are σ -bisimilar, for $\sigma = \emptyset$. Hence, if $L = \text{Log}(\mathcal{F})$ for a finite set of finite reflexive pointed frames and L has CIP, then $L = \text{Log}(\mathfrak{F})$ for a single finite pointed frame $\mathfrak{F}$.

We are now in a position to prove the following variant of Theorem 4.18 for Craig interpolant computation.

► **Theorem 5.3.** *Fix a tabular quasi-normal modal logic L with CIP and with $L = \text{Log}(\mathcal{F}_L)$. Let $\varphi \rightarrow \psi \in L$. If $\xi_{\mathfrak{F}} \in PL(\sigma_{\mathfrak{F}})$ are propositional Craig interpolants for $\text{tr}_{\mathfrak{F}}(\varphi) \rightarrow \text{tr}_{\mathfrak{F}}(\psi)$, $\mathfrak{F} \in \mathcal{F}_L$, then $\bigvee_{\mathfrak{F} \in \mathcal{F}_L} \text{rt}_{\mathfrak{F}, L}(\xi_{\mathfrak{F}})$ is a Craig interpolant for $\varphi \rightarrow \psi$ in L .*

Proof. Assume $\varphi \rightarrow \psi \in L$ is given. We show that (i) $\varphi \rightarrow \bigvee_{\mathfrak{F} \in \mathcal{F}_L} \text{rt}_{\mathfrak{F}, L}(\xi_{\mathfrak{F}}) \in L$ and (ii) $\bigvee_{\mathfrak{F} \in \mathcal{F}_L} \text{rt}_{\mathfrak{F}, L}(\xi_{\mathfrak{F}}) \rightarrow \psi \in L$. (i) can be shown in exactly the same way as in the proof of Theorem 4.18. To prove (ii), assume $\mathfrak{M}, 0 \models \bigvee_{\mathfrak{F} \in \mathcal{F}_L} \text{rt}_{\mathfrak{F}, L}(\xi_{\mathfrak{F}})$ with $\mathfrak{M}$ based on $\mathfrak{F}_i$. We show that $\mathfrak{M}, 0 \models \psi$. Take $\mathfrak{F}_j$ with $\mathfrak{M}, 0 \models \text{rt}_{\mathfrak{F}_j, L}(\xi_{\mathfrak{F}_j})$. By Proposition 4.14, there is a σ -bisimilar $\mathfrak{M}', 0$ based on $\mathfrak{F}_j$ with $v_{\mathfrak{M}'} \models \xi_{\mathfrak{F}_j}$. By Theorem 5.1, $i = j$ and we have an isomorphism f between the σ -reducts of $(\mathfrak{M}, 0)$ and $(\mathfrak{M}', 0)$. We modify $\mathfrak{M}'$ in the obvious way to obtain a model $\mathfrak{M}''$ such that f is an isomorphism from $(\mathfrak{M}, 0)$ onto $(\mathfrak{M}'', 0)$. Then $v_{\mathfrak{M}''} \models \xi_{\mathfrak{F}_i}$ since $v_{\mathfrak{M}'}$ and $v_{\mathfrak{M}''}$ coincide on $\sigma_{\mathfrak{F}_i}$. We obtain $v_{\mathfrak{M}''} \models \text{tr}_{\mathfrak{F}_i}(\psi)$ since $\xi_{\mathfrak{F}_i}$ is a Craig interpolant for $\text{tr}_{\mathfrak{F}_i}(\varphi) \rightarrow \text{tr}_{\mathfrak{F}_i}(\psi)$. Hence $\mathfrak{M}'', 0 \models \psi$. But then $\mathfrak{M}, 0 \models \psi$ since $(\mathfrak{M}, 0)$ and $(\mathfrak{M}'', 0)$ are isomorphic. ◀

Similarly to the computation of strongest implicants, also Theorem 5.3 can be strengthened by showing that to compute modal Craig interpolants only one query to an oracle computing propositional Craig interpolants is needed.

► **Theorem 1.3.** *Let L be a tabular quasi-normal modal logic that has CIP. Given a valid modal implication $\varphi \rightarrow \psi$ in L , and given a propositional Craig interpolant χ for $\text{tr}_L(\varphi) \rightarrow \text{tr}_L(\psi)$, it holds that $\text{rt}_L(\chi)$ is a Craig interpolant for $\varphi \rightarrow \psi$ with respect to L .*

The proof is similar to the proof of Theorem 1.1 (now using Theorem 5.3) and is therefore omitted.

We briefly discuss what happens for logics L without CIP. Clearly, in this case there must be $\varphi \rightarrow \psi \in L$ such that for any propositional Craig interpolant ξ for $\text{tr}_L(\varphi) \rightarrow \text{tr}_L(\psi)$ the modal formula $\text{rt}_L(\xi)$ is not a Craig interpolant for φ, ψ in L . For instance, consider $L_{\mathfrak{G}} = \text{Log}(\{(\mathfrak{G}, 0)\})$ defined in Example 3.2. Take the formulas

$$\gamma_0 = (\Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q)), \quad \gamma_1 = (\Diamond(\neg p \wedge r) \rightarrow \Box(\neg p \rightarrow r))$$

and let $\sigma = \{p\}$. Then the propositional formula ξ stating that “at least two successors satisfy p ” is (up to logical equivalence) the propositional uniform $\sigma_{\mathfrak{G}}$ -interpolant for $\text{tr}_{\mathfrak{G}, 0}(\gamma_0)$ and also the unique (up to logical equivalence) Craig interpolant for $\text{tr}_{\mathfrak{G}, 0}(\gamma_0) \rightarrow \text{tr}_{\mathfrak{G}, 0}(\gamma_1)$. The backward translation $\text{rt}_{\mathfrak{G}, 0}(\xi)$ is equivalent to $\Diamond p$. It is a strongest $L_{\mathfrak{G}}(\sigma)$ -implicate of γ_0 but neither a uniform $L_{\mathfrak{G}}(\sigma)$ -interpolant for γ_0 nor a Craig interpolant for $\gamma_0 \rightarrow \gamma_1$ in $L_{\mathfrak{G}}$.

There has recently been work on the computation of Craig interpolants for logics without CIP [3, 22]. Even for a tabular logic without CIP, it is of interest to compute, given $\varphi \rightarrow \psi \in L$ such that there exists a Craig interpolant for φ, ψ in L , such a Craig interpolant from any Craig interpolant for $\text{tr}_L(\varphi) \rightarrow \text{tr}_L(\psi)$ in polynomial time. We next show that this is *impossible* if there are valid implications in propositional logic for which all Craig

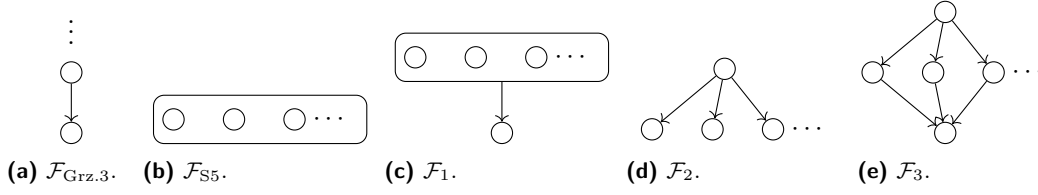


Figure 2 The finite rooted frames for the five pre-tabular normal logics containing S4. A boundary around a set of worlds indicates a *cluster* of worlds with wRw' for all w, w' in the cluster. Reflexive and transitive edges are omitted for clarity.

interpolants are of superpolynomial size. Assume $\varphi_n \rightarrow \psi_n$, $n \geq 0$, is a family of propositional formulas such that all Craig interpolants for $\varphi_n \rightarrow \psi_n$ are of superpolynomial size. Now consider again the tabular logic $L_{\mathfrak{S}}$. Let σ_n be the signature of φ_n, ψ_n . Let

$$\varphi'_n = \gamma_0 \wedge \Box \varphi_n, \quad \psi'_n = \gamma_1 \vee \Box \psi_n$$

with γ_0, γ_1 defined above. We assume that $\{p, q, r\}$ is disjoint from σ_n . The following statements hold:

1. there are propositional Craig interpolants of polynomial size for $tr_{\mathfrak{S},0}(\varphi'_n) \rightarrow tr_{\mathfrak{S},0}(\psi'_n)$: simply take the Craig interpolant for $tr_{\mathfrak{S},0}(\gamma_0) \rightarrow tr_{\mathfrak{S},0}(\gamma_1)$ defined above;
2. there are Craig interpolants for $\varphi'_n \rightarrow \psi'_n$ in $L_{\mathfrak{S}}$: take $\Box \chi_n$ with χ_n a propositional Craig interpolant for $\varphi_n \rightarrow \psi_n$; and
3. Craig interpolants for $\varphi'_n \rightarrow \psi'_n$ in $L_{\mathfrak{S}}$ are of superpolynomial size: from Craig interpolants for $\varphi'_n \rightarrow \psi'_n$ in $L_{\mathfrak{S}}$ one can construct in polynomial time in a straightforward way propositional Craig interpolants for $\varphi_n \rightarrow \psi_n$.

Of course, this result does not refute the possibility that a different translation can be used to polynomially reduce Craig interpolants for modal logics without CIP to propositional Craig interpolants.

6 Lower Bounds

Our aim in this section is to prove exponential lower bounds on the size of strongest implicates and Craig interpolants, for large classes of non-tabular modal logics. Our main result is as follows.

► **Theorem 1.5.** *Let L be any non-tabular normal modal logic that contains or is contained in $S4$ or the Gödel-Löb provability logic GL , or, more generally, has a non-tabular intersection with $S4$ or GL .*

1. *There exist modal formulas $(\varphi_n)_{n=1,2,\dots}$ of size polynomial in n and signatures $(\sigma_n)_{n=1,2,\dots}$ with $\sigma_n \subseteq \text{sig}(\varphi_n)$, such that φ_n has a strongest $L(\sigma_n)$ -implicate, and every strongest $L(\sigma_n)$ -implicate of φ_n has size at least 2^n .*
2. *There are modal formulas $(\varphi_n)_{n=1,2,\dots}$ and $(\psi_n)_{n=1,2,\dots}$ of size polynomial in n such that $\varphi_n \rightarrow \psi_n \in L$ for all $n \geq 1$, a Craig interpolant for $\varphi_n \rightarrow \psi_n$ exists for all $n \geq 1$, and every Craig interpolant for $\varphi_n \rightarrow \psi_n$ in L has size at least 2^n .*

While for Craig interpolants this theorem fully captures our results, for strongest implicates Part 1 follows from a significantly stronger result based on sufficient model-theoretic conditions for exponential size implicates. These conditions are defined next. Recall that a quasi-normal modal logic L has the *poly-size model property* if for every formula $\varphi \notin L$ there exists a rooted frame $(\mathfrak{F}, w)$ validating L and refuting φ which is of polynomial size in $|\varphi|$. We say

that L has the *exponential growth property* if there is a polynomial function f such that for every $n > 0$ there exists a rooted frame $(\mathfrak{F}, w)$ validating L such that at least 2^n worlds are reachable from w by an R -path of length at most $f(n)$ starting at w . A few standard modal logics such as S5, GL.3, and Grz.3 have both the poly-size model property and exponential growth property. We are in a position now to formulate a sufficient condition for exponential size strongest implicates.

► **Theorem 6.1.** *Let L be a quasi-normal modal logic contained in a quasi-normal modal logic L' with the poly-size model property and the exponential growth property. Then there exist modal formulas φ_n and signatures $\sigma_n \subseteq \text{sig}(\varphi_n)$ with $|\varphi_n|$ polynomial in n such that the strongest $L(\sigma_n)$ -implicates of φ_n exist and are of size $\geq 2^n$.*

Proof. By the poly-size model property of L' , we can take $k > 0$ such that satisfiable formulas of size m are satisfied in models of size at most m^k , for sufficiently large m . We also take a polynomial function f witnessing the exponential growth property of L' . Consider atoms $\sigma_n = \{p_0, \dots, p_{kn-1}\}$ and let $q_0, \dots, q_{kn-1}$ be a set of atoms disjoint from σ_n . Let

$$\begin{aligned} \varphi_n = & \Box^{\leq f(kn)} \bigvee_{i=0}^{kn-1} \neg(p_i \leftrightarrow q_i) \wedge \\ & \Box^{\leq f(kn)-1} \bigwedge_{i=0}^{kn-1} ((q_i \rightarrow \Box q_i) \wedge (\neg q_i \rightarrow \Box \neg q_i)) \end{aligned}$$

The formula $\chi_n = \bigvee_{t \in T_{kn}} \Box^{\leq f(kn)} \neg t$ is then easily seen to be a strongest $L(\sigma_n)$ -implicate of φ_n , where T_{kn} is the set of types $\bigwedge_{i=0}^{kn-1} l_i$ with $l_i \in \{p_i, \neg p_i\}$ for $i = 0, \dots, kn-1$. Assume now for a proof by contradiction that χ_n is equivalent on frames validating L to a formula ψ_n of size $< 2^n$, for sufficiently large n . Then this is also the case for $L' \supseteq L$. Take $\neg\psi_n$ which states that all 2^{kn} types in T_{kn} are satisfied in worlds reachable in $f(kn)$ steps. By the exponential growth property of L' , $\neg\psi_n$ is satisfiable in an L' -frame. We derive a contradiction since then $\neg\psi_n$ is satisfied in a model of size $< 2^{kn}$, for sufficiently large n . ◀

We next spell out the consequences of Theorem 6.1 and show that the first part of Theorem 1.5 follows. Observe first that every non-tabular quasi-normal modal logic containing K4 has the exponential growth property since every world reachable from a world is reachable in one step already. Hence, above K4, it suffices to analyze the poly-size model property. Call a normal modal logic L *pre-tabular* if every normal modal logic properly containing L is tabular.² Pre-tabular logics have been investigated extensively in modal logic as they separate tabular from non-tabular logics. For the following result and further details on pre-tabular logics, we refer the reader to [8, Chapter 12].

► **Proposition 6.2.**

- (1) *Every non-tabular normal modal logic is contained in a pre-tabular normal modal logic.*
- (2) *All normal modal logics containing a tabular normal modal logic are tabular.*
- (3) *There are five pre-tabular normal modal logics containing $S4$. These are $L(\mathcal{F})$ for $\mathcal{F}$ depicted in Figure 2.*
- (4) *There are countably many pre-tabular normal modal logics containing GL . These are $L(\mathcal{F})$ for $\mathcal{F}$ depicted in Figure 3.*

² We use the term pre-tabular logic only in the class of *normal* modal logics. Pre-tabular logics have also been studied in the class of quasi-normal modal logics [8]. The two notions are indeed different. We focus on normal ones for simplicity here.

Observe that $\text{Grz.3} = L(\mathcal{F}_{\text{Grz.3}})$, $\text{S5} = L(\mathcal{F}_{\text{S5}})$ are pre-tabular logics containing S4 and $\text{GL.3} = L(\mathcal{F}_{\text{GL.3}})$ is a pre-tabular logic containing GL. Now the first half of Theorem 1.5 follows from Theorem 6.1, Proposition 6.2 (1), and the following result.

► **Proposition 6.3.** *All pre-tabular normal modal logics containing S4 or GL have both the poly-size model property and, trivially, the exponential growth property.*

The proof of Proposition 6.3 is straightforward, given the description of frames for pre-tabular normal modal logics in Figures 2 and 3. In fact, to construct poly-size satisfying models one can use selective filtration in the same way as in the proof of the second half of Theorem 1.5 below.

We come to the proof of Theorem 1.5 (2). We define modal formulas $(\varphi_n)_{n=1,2,\dots}$ and $(\psi_n)_{n=1,2,\dots}$ of size polynomial in n and show for each pre-tabular normal modal logic L' containing S4 or GL separately that for every normal modal logic L contained in L' Craig interpolants for $\varphi_n \rightarrow \psi_n$ exist but are of exponential size.

The proof requires slightly different arguments for each pre-tabular normal modal logic. Consider $\sigma_n = \{r\} \cup \{p_0, \dots, p_{n-1}\}$ and let T_n denote the set of types $\bigwedge_{i=0}^{n-1} l_i$ with $l_i \in \{p_i, \neg p_i\}$ for $i = 0, \dots, n-1$. Let $q_0, \dots, q_{n-1}, q'_0, \dots, q'_{n-1}$ be a set of atoms disjoint from σ_n . We aim to enforce the Craig interpolants $I_n = \bigvee_{t \in T_n} (r \wedge t \wedge \Diamond(\neg r \wedge t))$. To this end, define

$$\chi(p_0, \dots, p_{n-1}, q_0, \dots, q_{n-1}) = \bigwedge_{0 \leq i < n} ((p_i \leftrightarrow q_i) \wedge (q_i \rightarrow \Box q_i) \wedge (\neg q_i \rightarrow \Box \neg q_i))$$

and obtain $\chi(p_0, \dots, p_{n-1}, q'_0, \dots, q'_{n-1})$ from $\chi(p_0, \dots, p_{n-1}, q_0, \dots, q_{n-1})$ by replacing q_i by q'_i , for all $i < n$. Then let

$$\varphi_n = r \wedge \chi(p_0, \dots, p_{n-1}, q_0, \dots, q_{n-1}) \wedge \Diamond(\neg r \wedge \bigwedge_{0 \leq i < n} (p_i \leftrightarrow q_i))$$

and

$$\psi_n = r \wedge (\chi(p_0, \dots, p_{n-1}, q'_0, \dots, q'_{n-1}) \rightarrow \Diamond(\neg r \wedge \bigwedge_{0 \leq i < n} (p_i \leftrightarrow q'_i)))$$

Observe that for every pointed frame $\mathfrak{F}, 0$ we have $\mathfrak{F}, 0 \models \varphi_n \rightarrow \psi_n$ and I_n is a Craig interpolant for $\varphi_n \rightarrow \psi_n$ for every quasi-normal modal logic. The following observation follows directly from the fact that every model of I_n can be expanded to a model of φ_n and every model of $\neg I_n$ can be expanded to a model of $\neg \psi_n$.

Fact 1. If L is a quasi-normal modal logic, then every Craig interpolant for $\varphi_n \rightarrow \psi_n$ is equivalent to I_n in L .

We now argue that for every L contained in a normal pre-tabular logic containing S4 or GL, there is no formula equivalent to I_n on frames validating L of size $< 2^n$. We show this for $L \subseteq \text{Log}(\mathcal{F}_{\text{Grz.3}})$ and for $L \subseteq \text{Log}(\mathcal{F}_{\text{S5}})$, for the remaining pre-tabular logics it can be shown in a similar manner.

Given any transitive model $\mathfrak{M} = (\mathfrak{F}, R, V)$, a world $w \in W$ is called *maximal for a formula* χ in $\mathfrak{M}$ if $\mathfrak{M}, w \models \chi$ and for any w' with $\mathfrak{M}, w' \models \chi$ and wRw' it follows that $w' = w$ or $w'Rw$.

Case 1. Let $L \subseteq \text{Log}(\mathcal{F}_{\text{Grz.3}})$. Assume for a proof by contradiction that χ_n is equivalent to I_n in L and of size $< 2^n$.

We construct a model $\mathfrak{M}_{1,n}$ based on $\mathfrak{F}_{1,n} = (W_{1,n}, R_{1,n})$ as follows: let $W_{1,n} = M_{1,n} \cup N_{1,n}$ with $M_{1,n} = \{a_0, \dots, a_{2^n-1}\}$ and $N_{1,n} = \{b_0, \dots, b_{2^n-1}\}$. Define $R_{1,n}$ as the reflexive and transitive closure of

$$\{(a_i, b_0) \mid i < 2^n\} \cup \{(b_i, b_{i+1}) \mid i < 2^n - 1\}.$$

Then $\mathfrak{F}_{1,n}$ validates L . Let $\mathfrak{M}_{1,n}$ be defined in such a way that

- every type $t \in T_n$ is satisfied in exactly one a_i , $i < 2^n$;
- r is satisfied in a_i for all $i < 2^n$ and not in any b_i with $i < 2^n$;
- every type $t \in T_n$ is satisfied in exactly one b_i , $i < 2^n$.

We have $\mathfrak{M}_{1,n}, a_i \models I_n$ for all $i < 2^n$. Hence we have $\mathfrak{M}_{1,n}, a_i \models \chi_n$. Pick for every $\xi \in \text{sub}(\chi_n)$ satisfied in some world in $N_{1,n}$ a maximal world b_ξ in $N_{1,n}$ satisfying ξ in $\mathfrak{M}_{1,n}$. For each $i < 2^n$, let $\mathfrak{M}_i$ denote the restriction of $\mathfrak{M}_{1,n}$ to $\{a_i\} \cup \{b_\xi \mid b_\xi \in \text{sub}(\chi_n) \text{ picked}\}$. Then each pointed $\mathfrak{M}_i, c$ with c in $\mathfrak{M}_i$ satisfies the same type $t \in T_n$ as $\mathfrak{M}_{1,n}, c$. Moreover, $\mathfrak{M}_i, a_i \models \chi_n$ for all $i < 2^n$ can be shown by induction. Hence $\mathfrak{M}_i, a_i \models I_n$ since the $\mathfrak{M}_i, a_i$ are based on frames validating L . However, note that there is a type in T_n that is not satisfied in any b_ξ since the number of b_ξ in $\mathfrak{M}_i$ is $< 2^n$. Hence there is an a_i with $\mathfrak{M}_i, a_i \not\models I_n$ and we have derived a contradiction.

Case 2. $L \subseteq \text{Log}(\mathcal{F}_{S5})$. Assume for a proof by contradiction that χ_n is equivalent to I_n in L and of size $< 2^n$.

We construct a model $\mathfrak{M}_{2,n}$ based on $\mathfrak{F}_{2,n} = (W_{2,n}, R_{2,n})$ as follows. Define $W_{2,n} = M_{2,n} \cup N_{2,n}$ with $M_{2,n} = \{a_0, \dots, a_{2^n-1}\}$ and $N_{2,n} = \{b_0, \dots, b_{2^n-1}\}$ and let $R_{2,n} = W_{2,n} \times W_{2,n}$. Then $\mathfrak{F}_{2,n}$ validates L . Let the valuation of $\mathfrak{M}_{2,n}$ be defined in the same way as the valuation of $\mathfrak{M}_{1,n}$ in Case 1.

We have $\mathfrak{M}_{2,n}, a_i \models I_n$ for all $i < 2^n$. Hence we have $\mathfrak{M}_{2,n}, a_i \models \chi_n$. Pick for every $\xi \in \text{sub}(\chi_n)$ satisfied in some world in $N_{2,n}$ a world b_ξ in $N_{2,n}$ satisfying ξ in $\mathfrak{M}_{2,n}$. Also pick for every $\xi \in \text{sub}(\chi_n)$ satisfied in some world in $M_{2,n}$ a world a_ξ in $M_{2,n}$ satisfying ξ in $\mathfrak{M}_{2,n}$. Let $\mathfrak{M}_i$ denote the restriction of $\mathfrak{M}_{2,n}$ to

$$\{a_i\} \cup \{b_\xi \mid b_\xi \in \text{sub}(\chi_n) \text{ picked}\} \cup \{a_\xi \mid a_\xi \in \text{sub}(\chi_n) \text{ picked}\}.$$

Now the proof continues as in Case 1: Clearly each $\mathfrak{M}_i, c$ with c in $\mathfrak{M}_i$ satisfies the same type $t \in T_n$ as $\mathfrak{M}_{2,n}, c$ and it is easy to see that $\mathfrak{M}_i, a_i \models \chi_n$ for all $i < 2^n$. Hence $\mathfrak{M}_i, a_i \models I_n$ for $i < 2^n$. However, there is a type in T_n that is not satisfied in any b_ξ since the number of b_ξ in $\mathfrak{M}_i$ is $< 2^n$. Hence there is an a_i with $\mathfrak{M}_i, a_i \not\models I_n$ and we have derived a contradiction. We omit the remaining cases here.

The logic Alt_1 can be used to show that there are non-tabular normal modal logics for which an exponential lower bound for the size of uniform interpolants would imply $P \neq NP$. The proof is again by reduction to uniform interpolants for propositional logic. Observe that Alt_1 has the poly-size model property but does not have the exponential growth property.

► **Theorem 6.4.** *Let σ be a signature, and $\hat{\sigma} = \sigma \times \mathbb{N}$. There are poly-time translations*

$$tr_{alt} : ML(\sigma) \rightarrow PL(\hat{\sigma}) \quad \text{and} \quad rt_{alt} : PL(\hat{\sigma}) \rightarrow ML(\sigma)$$

such that for every modal formula φ , every $\tau \subseteq \sigma$ and every uniform $\tau \times \mathbb{N}$ -interpolant ψ for $tr_{alt}(\varphi)$ in propositional logic, $rt_{alt}(\psi)$ is a uniform $\text{Alt}_1(\tau)$ -interpolant for φ .

Here, if $\varphi \in ML(\sigma)$ has modal depth n , then $tr_{alt}(\varphi) \in PL(\sigma \times \{0, \dots, n\})$.

Proof. We adapt the translation developed in Section 4. However, naively applying the method from Section 4, would result in an exponentially sized interpolant. We therefore define a simpler translation that relies on the fact that one can work with finite frames for Alt_1 without cycles and that on such frames bisimulations are trivial: between any two rooted models there is at most one bisimulation and, if it exists, it is an isomorphism.

Let $\varphi \in \text{ML}(\sigma)$ have modal depth n and assume $\tau \subseteq \sigma$. The translation $tr_{\text{alt}}(\varphi)$ will only be concerned with the frames $\mathcal{F}_n = \{\mathfrak{F}_0, \dots, \mathfrak{F}_n\}$ where $\mathfrak{F}_k = (W_k, R_k)$ is given by $W_k = \{k, \dots, 0\}$, $R_k = \{(i, i-1) \mid 0 < i \leq k\}$ and the root of $\mathfrak{F}_i$ is the world i (so here we do not follow the convention that the root is labeled 0). In other words, $\mathcal{F}_n$ is the set of linear chains of length up to n , with the worlds counting down from k . We adapt the technique developed in Section 4 to find a strongest $\text{Alt}_1(\tau)$ -implicate which, because Alt_1 has CIP, will also be a uniform $\text{Alt}_1(\tau)$ -interpolant.

Let $\mathfrak{F}_i \in \mathcal{F}_n$, and let $\mathfrak{M}$ be any model based on $\mathfrak{F}_i$. We define the associated propositional model $v_{\mathfrak{M}}$ over $\sigma \times \{0, \dots, i\}$ by taking $v_{\mathfrak{M}}(p_j) = 1$ iff $\mathfrak{M}, j \models p$. Additionally, we recursively define a set of translations $tr_i : \text{ML}(\sigma) \rightarrow \text{PL}(\hat{\sigma})$, for $i \in \mathbb{N}$, by

$$\begin{aligned} tr_i(p) &= p_i, \\ tr_i(\neg\varphi) &= \neg tr_i(\varphi), \\ tr_i(\varphi_1 \vee \varphi_2) &= tr_i(\varphi_1) \vee tr_i(\varphi_2), \\ tr_i(\Box\varphi) &= \begin{cases} \top & \text{if } i = 0 \\ tr_{i-1}(\varphi) & \text{if } i > 0 \end{cases} \end{aligned}$$

and a reverse translation $rt_i : \text{PL}(\sigma \times \{0, \dots, i\}) \rightarrow \text{ML}(\sigma)$ by

$$rt_i(\psi) = \psi[p_j \mapsto \Diamond^{i-j}p].$$

It is straightforward to verify that for every $\chi \in \text{ML}(\sigma)$, $v_{\mathfrak{M}} \models tr_i(\chi)$ if, and only if, $\mathfrak{M}, i \models \chi$, and that for every propositional formula $\psi \in \text{PL}(\sigma \times \{0, \dots, i\})$, $v_{\mathfrak{M}} \models \psi$ if, and only if, $\mathfrak{M}, i \models rt_i(\psi)$.

We next define a strongest $\text{Alt}_1(\tau)$ -implicate for φ using uniform propositional $\tau \times \{0, \dots, i\}$ -interpolants for $tr_i(\varphi)$, for $0 \leq i \leq n$. Let, for each $0 \leq i \leq n$, ψ_i be a uniform propositional $\tau \times \{0, \dots, i\}$ -interpolant for $tr_i(\varphi)$. Consider the modal formula

$$\psi = \bigwedge_{0 \leq i \leq n-1} (\Diamond^i \Box \perp \rightarrow \psi_i[p_j \mapsto \Diamond^{i-j}p]) \wedge (\Diamond^n \top \rightarrow \psi_n[p_j \mapsto \Diamond^{n-j}p])$$

which uses propositional variables in τ only. It is straightforward to show that ψ is a strongest $\text{Alt}_1(\tau)$ -implicate for φ , as required.

We have used multiple translations tr_i and rt_i and also multiple uniform propositional interpolants, as was previously done for tabular logics determined by sets of finite frames in Section 4.2. A single translation tr_{alt} and single converse translation rt_{alt} are obtained in the same way as in Section 4.3. $\blacktriangleleft$

We note that it is also open whether the dichotomy above can be extended to K4 since the pre-tabular normal logics containing K4 are significantly more complex than the pre-tabular normal modal logics containing S4 or GL. For instance, there are pre-tabular normal modal logics containing K4 without the poly-size model property [8, Chapter 12].

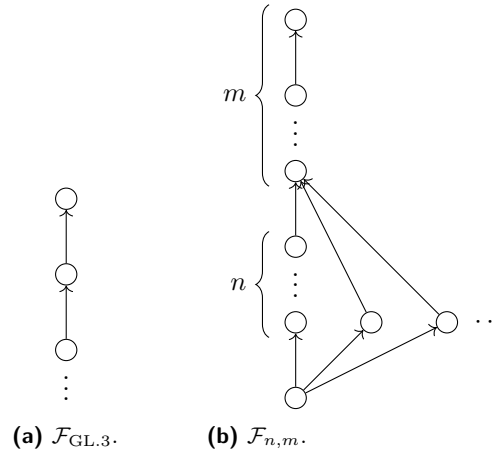


Figure 3 The finite rooted frames for the pre-tabular normal logics containing GL. Transitive edges are omitted for clarity.

7 Conclusion

We have investigated size bounds for Craig interpolants, uniform interpolants, and strongest implicates, for various families of normal modal logics, and, more generally, quasi-normal modal logics.

For *tabular* quasi-normal modal logics, our results show that size upper bounds for strongest implicates are essentially the same as in the case of propositional logic (modulo polynomial translations). This is, in some sense, the best one can hope for: a polynomial upper bound would imply that $\text{NP} \subseteq \text{P/poly}$, a longstanding open problem in complexity theory, whereas a super-polynomial lower bound would imply that $\text{NP} \not\subseteq \text{P/poly}$. We obtained analogous results regarding Craig interpolants and uniform interpolants (for tabular quasi-normal modal logics that have CIP or, equivalently, UIP).

Complementing these results for tabular logics, we established exponential lower bounds on the size of strongest implicates, uniform interpolants, and Craig interpolants, which apply to all *non-tabular* normal logics that are contained in or contain S4 or GL (or, more generally, that have a non-tabular intersection with S4 or GL). This covers most non-tabular normal modal logics in practice, and matches known single-exponential upper bounds on the size of Craig interpolants for common modal logics such as K, K4, GL, S5, and Grz, as well as known single-exponential upper bounds on the size of uniform interpolants for S5 and K. The problem of finding tight bounds on the size of uniform interpolants for GL and Grz remains open.

The main problem left open is whether one can establish a dichotomy of the following form for all (quasi-)normal modal logics: either computing strongest implicates is poly-time reducible to computing uniform interpolants in classical propositional logic or strongest implicates are of at least exponential size, and similarly for Craig interpolants for modal logics having CIP. The logic Alt_1 shows that to prove this, one has to replace tabularity by some weaker condition. For normal modal logics containing K4, however, we conjecture that tabularity can still be used to show a dichotomy. Another intriguing problem is the size of Craig interpolants (when they exist) for tabular modal logics without CIP. We have seen that our current reduction does not work in this case, but there could still be a completely different approach. On the other hand, it is known that Craig interpolants often become much harder to compute for logics without CIP (and uniform interpolants for logics without UIP) [17, 16, 15].

As discussed earlier, these results all pertain to formula-size as measured based on a dag representation. Interestingly, it is not difficult to show that our reduction of strongest implicates in tabular quasi-normal modal logics to uniform interpolants in propositional logic (and, similarly, of Craig interpolants for tabular logics with CIP) provides “propositional sized” implicates also if a standard tree-representation is used. In fact, while the tree-size of the “forward-translation” $tr_L(\varphi)$ of a formula φ can be exponential, the “backward-translation” $rt_L(\varphi)$ is always of polynomial tree-size. The latter suffices to obtain strongest implicates of polynomial tree-size in the size of the propositional uniform interpolant and $tr_L(\varphi)$ can always be represented, using additional propositional variables, as a formula of polynomial tree-size. This observation regarding rt_L can be used to lift further properties from propositional logic to tabular modal logics. For instance, assume that every propositional formula of dag-size n is equivalent to a propositional formula ψ of tree-size $f(n)$. Then, in tabular modal logics, every modal formula of dag-size n is equivalent to a modal formula of tree-size $poly(f(poly(n)))$.

Note that, for the modal logic K (which has UIP), it is known that the tree-size of uniform interpolants (and hence also strongest σ -implicates and Craig interpolants) can be bounded by an exponential function in the tree-size of input formula(s). On the other hand, it remains an open problem whether the exponential upper bounds for Craig interpolants in K4, S4, GL, and Grz still hold if one uses a tree representation of interpolants.

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