The ∞-Category of ∞-Categories in Simplicial Type Theory

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Abstract

Simplicial type theory (STT) was introduced by Riehl and Shulman to leverage homotopy type theory to prove results about $(\infty, 1)$-categories. Initial work on simplicial type theory focused on "formal" arguments in higher category theory and, in particular, no non-trivial examples of ∞-category theory were constructible within STT. More recent work has changed this state of affairs by applying techniques developed initially for cubical type theory to construct the ∞-category of spaces. We complete this process by constructing the ∞-category of ∞-categories, recovering one of the main foundational results of ∞-category theory (straightening–unstraightening) purely type-theoretically. We also show how this construction enables new examples of the directed version of the structure identity principle: the structure homomorphism principle.

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1 Introduction

A defining characteristic of dependent type theories is their focus on universes of (small) types. More than in other foundations of mathematics, such universes are critical for even proving such basic properties as $0 \neq 1$ [38]. This focus on universes is only intensified with *homotopy type theory* (HoTT) [39] where the universe is supplemented with the univalence axiom. Such *univalent* universes allow type theorists to view types as a synthetic incarnation of spaces (i.e., ∞-groupoids) with the intensional identity type $a =_A b$ modeling paths in the type A. The utility of this viewpoint is demonstrated by the development of synthetic homotopy theory inside of HoTT: a reconstruction of classical results in homotopy theory with simpler and more conceptual proofs.

A long-standing challenge in HoTT has been to broaden the reach of synthetic homotopy theory to include homotopy-coherent algebraic structures and, especially, the homotopical enhancement of category theory: $(\infty, 1)$-category theory.[1] This problem has motivated numerous different proposals and approaches [15, 40, 26, 32, 25, 13, 27, 41, 1, 42, 24, 23]. We emphasize that this research is not motivated purely by internal considerations with HoTT; it also comes from the need for a version of ∞-category theory which avoids some of the bureaucracy currently necessary with present approaches to the subject [30]. In particular, as ∞-categories see more use in algebraic geometry, homotopy theory, and algebraic K theory among other fields, more and more developments are carried out with an informal version of ∞-category theory, separate from the more bureaucratic quasicategorical foundations established by Joyal and Lurie [12, 17]. These arguments similar to proofs in homotopy theory using HoTT, which avoid explicitly discussing e.g., topological spaces or Kan complexes. It is therefore conjectured that an extension of HoTT which can express ∞-category theory would similarly provide users with the desired language for informal ∞-category theory.

While many different approaches to such an extension of HoTT have been proposed, we will focus on the approach introduced by [32]. There the authors leveraged a non-standard model of HoTT where types are realized by simplicial spaces. In particular, they showed that the complete Segal spaces – a known model of ∞-categories [29] – then arise as certain types satisfying a pair of easily-defined properties. Thus, in this setting not every type is an ∞-category, but every ∞-category gives rise to a valid type.

Concretely, *simplicial type theory* extends HoTT with a *directed interval*, a postulated totally ordered lattice $(\mathbb{I}, 0, 1, \leq)$. This new type is meant to represent the ∞-category with two objects $0, 1$ and a single non-identity morphism $0 \to 1$ – an interpretation justified by the model of STT in simplicial spaces – and we then use $\mathbb{I}$ to define morphisms in an arbitrary type A as ordinary functions $\mathbb{I} \to A$. By constraining the endpoints of a synthetic morphism, we arrive at the definition of the space of synthetic morphisms in a type: $\hom_A(a, b) = \sum_{f:\mathbb{I}\to A} f\,0 = a \times f\,1 = b$.

[32] then demonstrate that the definition of an ∞-category can be formulated concisely as a *predicate* isCat on types, essentially requiring every pair of composable morphisms have a unique composite. Furthermore, they show that ordinary functions between such types constitute functors and that other classical definitions in ∞-category theory become expressible. Subsequent work has further expanded this approach, developing fibered category theory [4, 43], limits and colimits [3], etc.

While not every type constitutes an ∞-category in STT, many type-theoretic operations preserve the property of being an ∞-category. For instance, $\mathbf{0}$ and $\mathbf{1}$ are the initial and terminal categories, $A \times B$ $(A + B)$ is the (co)product category, and $A \to B$ is the category of functors. As an extension of HoTT, STT comes equipped with a (hierarchy of) universes and it is therefore natural to ask:

Is $\mathcal{U}$ a recognizable category, e.g., the category of categories?

Unfortunately, the answer is negative; $\mathcal{U}$ is the canonical example of a type that is *not* an ∞-category in STT. Moreover, simple subtypes of the universe (e.g., $\sum_{A:\mathcal{U}} \mathsf{isCat}(A)$) also

[1] In both the title and hereafter, we shall write "∞-category" or even "category" to refer to $(\infty, 1)$-categories. If we wish to single out ordinary categories, we shall specifically denote them by 1-categories.

do not yield categories as the synthetic morphisms $\mathbb{I} \to \sum_{A:\mathcal{U}} \mathsf{isCat}(A)$ neither compose nor faithfully represent functors. However, it has long been conjectured that the category of categories should be constructible in STT as a certain subtype of the universe.

We address this final gap in the foundations of STT by proving this conjecture and constructing the category of categories as a subtype $\mathsf{Cat} \hookrightarrow \mathcal{U}$ and verifying its core properties. Once more, this addresses a very real need for ∞-categorical practioners: no synthetic theory of categories can develop serious mathematics while lacking such a basic construction as the category of categories.

1.1 Directed univalence and Cat

What criteria should be used to determine if a subtype $\mathsf{Cat} \hookrightarrow \mathcal{U}$ is a valid definition of the category of categories? If one is not considering ∞-categories, the answer to this question is straightforward: Cat is a valid definition if the objects denote precisely small types satisfying isCat, synthetic morphisms are exactly functions (i.e., functors) between these types, and the composition and identity operations behave as expected. In the ∞-categorical case, the story does not end here; we must also convince ourselves that all the higher synthetic morphisms also behave "as expected". However, it is far from clear what the expected behavior ought to be! Instead, we take a different approach by isolating a particular universal property for the embedding $\mathsf{Cat} \hookrightarrow \mathcal{U}$ and arguing that all the other properties of Cat stem from this single property.

The universal property in question is an ∞-categorical version of the Grothendieck correspondence, often referred to as straightening–unstraightening in the ∞-categorical context. Classically, the Grothendieck correspondence states there is an equivalence between pseudofunctors $\mathcal{C} \to \mathbf{Cat}$ for some 1-category $\mathcal{C}$ and pairs of a category $\mathcal{E}$ and a cocartesian fibration $\mathcal{E} \to \mathcal{C}$.[2] Naively, this suggests a simple universal property for the type Cat: We should require for every ∞-category C an equivalence between the type of functions $C \to \mathsf{Cat}$ and the type $\sum_{E:\mathcal{U}} \sum_{\pi:E\to C} \mathrm{isCocart}(\pi)$. Unfortunately, this definition is too straightforward: if such a Cat existed we would be able to show that isCocart held for every map π which is certainly not true.

The problem is familiar to cubical type theorists: we are essentially asking for Cat to be a *universal fibration*, where isCocart is our notion of fibrancy. It is well-known that one cannot give the universal property for the univalent universe in the extensional type theory of cubical sets [28]: the naive equivalence is only valid with respect to *closed* elements and cannot be internalized this way into type theory where it would also apply to elements in an arbitrary context. A solution to this problem in cubical type theory was presented by [16]. There the authors give a description of a universal fibration in an extensional type theory for cubical sets using *modal* type theory. They supplement type theory with an idempotent comonad $\flat$, such that the modal type $\langle \flat \mid A \rangle$ contains only the *global* elements of A. After further extending type theory with a right adjoint to $\mathbb{I} \to -$, they are then not only able to express the desired universal property (appropriately annotated with $\flat$), but also use the aforementioned right adjoint to derive it.

This idea has been adapted to variants of simplicial type theory first by [41] and later by [9] to construct a category of *discrete* ∞-categories (spaces). We follow on this line of work – specifically [9] – and use the $\flat$ modality to state the correct universal property for the category of categories and produce a type satisfying it. In particular:

[2] In the 1- and 2-categorical literature these are often referred to as Grothendieck opfibrations. We use "cocartesian" as it is the standard term in ∞-category theory.

▶ **Theorem 33.** Cat *is the base of the universal cocartesian family, i.e., for any* $C :_\flat \mathcal{U}_{\boxslash}$*, we have* $\langle \flat \mid C \to \mathsf{Cat}\rangle \simeq \left\langle \flat \mid \sum_{E:C\to\mathcal{U}} \mathrm{isCocart}(E)\right\rangle$.

On its own, this statement is not worth much. We do not know whether Cat is a category itself, let alone to what its objects and morphisms correspond. To resolve this, we give a detailed analysis of the structure of cocartesian fibrations over certain categories, relying on the established results in simplicial type theory. As a particular case of these results, we conclude that cocartesian fibrations over $\mathbb{I}$ are determined by (1) a pair of categories C, D along with (2) a function $C \to D$. Consequently, we are able to prove the directed univalence theorem for Cat:

▶ **Corollary 38** (Directed univalence)**.** *If* A, B *are* $\flat$*-elements of* Cat*, then there is an equivalence* $\mathsf{dua} : \hom_{\mathsf{Cat}}(A, B) \simeq \langle \flat \mid A \to B\rangle$.

Here we see the first major divergence between the case of categories and the previously considered case of discrete categories. Directed univalence for Cat only guarantees that $\hom_{\mathsf{Cat}}(A, B)$ is equivalent to $\langle \flat \mid A \to B\rangle$ whereas in the discrete case one obtains an equivalence with $A \to B$. This is inevitable: *no* category can have hom-spaces equivalent to $A \to B$ for arbitrary categories A, B. Indeed, one may prove that the synthetic morphisms of $\hom_{\mathsf{Cat}}(A, B)$ itself are necessarily invertible, but this is generally false for $A \to B$. However, the appearance of modalities in directed univalence is a significant complication.

Further consequences of our analysis prove that Cat is a category and that composition of synthetic morphisms corresponds (under directed univalence) to composition of ordinary functions. In total then, we show that the universal property of Cat is sufficient to derive all the expected behavior of a category of categories as well as uniquely characterize Cat itself.

Directed univalence opens up a wide array of applications. For instance, we use it to show that Cat contains a number of recognizable *reflective* subcategories (truncated categories, partial orders, spaces, etc.). It also allows us to build new and important categories by combining Cat with existing type-theoretic connectives. As a simple example, we show that $\sum_{c:\mathsf{Cat}} c$ can be analyzed using directed univalence and use its attendant *structure homomorphism principle* (SHP) to see that $\sum_{c:\mathsf{Cat}} c$ is the *lax* slice category $\mathbf{1} /\!\!/ \mathsf{Cat}$.

Finally, while Theorem 33 is a form of the Grothendieck correspondence, it is not the strongest form possible. This would state that there is an equivalence of ∞-*categories* between cocartesian families over C and the functor category $C \to \mathsf{Cat}$. We adapt an argument sketched by [5] to STT to show that Theorem 33 upgrades to an equivalence of categories and thereby give a synthetic proof of the *straightening-unstraightening* theorem [17].

1.2 Contributions and outline

Our main contribution is the first construction of a directed univalent category of categories within STT together with a verification of all its essential properties. This is the last major primitive missing from the theory of ∞-categories in simplicial type theory and, additionally, also constitutes a new and novel approach to a foundational theorem in ∞-category theory itself. With this in place, we also note that a putative full directed type theory can be interpreted into ∞-categories by giving a mere syntactic translation to the appropriate fragment of STT.

- We extend the methodology of [16] to construct a universal cocartesian family $\mathsf{Cat}_\bullet \to \mathsf{Cat}$.
- We use this universal property to show that Cat is a directed-univalent category
- We prove various properties of Cat and showcase the novel applications of SHP it enables.
- Finally, we give a fully synthetic proof of the straightening–unstraightening theorem.

Over all, we demonstrate how type theory (through STT) is highly effective for proving major results in higher category theory.

The remainder of the paper is organized as follows. In Section 2 we recall the main ideas of simplicial type theory, its triangulated type theory variant, and the modal extensions thereof. This, along with Section 3 on cocartesian fibrations, is intended to make the paper as self-contained as possible. In Section 4 we define Cat and prove Theorem 33, the main construction of this paper. In Section 5, we combine our new results on cocartesian fibrations along with the results of the prior section to prove Corollary 38 along with the fact that Cat is a category. Finally, in Sections 6 and 7, we give a number of consequences of our results, including a proof of straightening–unstraightening (Corollary 52).

2 Simplicial and triangulated type theory

We now turn to giving a more precise account of the flavor of simplicial type theory we shall use. There are two major modifications to the theory compared with the original system studied by [32]. First is that we supplement our theory with a collection of modalities, including the $\flat$ modality discussed in the introduction. We do this using MTT, a general purpose modal dependent type theory [8]. Secondly, the intended model of STT is simplicial spaces, the ∞-categorical version of $\mathbf{PSh}(\Delta)$. However, in order to leverage the techniques of [16] to define Cat, we require a postulated *amazing right adjoint* to $\mathbb{I} \to -$. Unfortunately, in the intended model of simplicial spaces, such a right adjoint simply does not exist; the category of simplices does not have products.

To circumvent this, [9] introduced a further relaxation of the modal variant of STT: *triangulated type theory* $\mathsf{TT}_{\boxslash}$. The intended model of $\mathsf{TT}_{\boxslash}$ is *Dedekind cubical* spaces, rather than simplicial ones. Simplicial spaces embed into this intended model as a subtopos and the original structure therefore remains. Concretely, this change involves relaxing the requirement that $\mathbb{I}$ be totally ordered to merely asking it to be a bounded distributive lattice. In this section, we recall the details of $\mathsf{TT}_{\boxslash}$ relevant for this work and, in particular, describe the modal extensions necessary to construct the category of categories.

2.1 First steps in triangulated type theory

As mentioned in Section 1, STT and its relaxation $\mathsf{TT}_{\boxslash}$, extend homotopy type theory. Accordingly, we begin by recalling the basic definitions and notations from homotopy type theory. For a more complete account, see the [39].

Homotopy type theory begins with intensional Martin-Löf type theory, complete with a hierarchy of universes, $\mathcal{U}_0, \mathcal{U}_1, \ldots$, and intensional identity types $a =_A b$. We use $\prod$ and $\sum$ for the standard dependent product and sum types of this type theory and use $p_!$ for the transport map $B(x) \to B(y)$ induced by $p : x = y$. The key extension of HoTT, the univalence axiom, governs the behavior of the intensional identity type of the universe. In particular, this axiom relates equality in the universe to *equivalences*. See op. cit. for a detailed discussion of the type of equivalences $A \simeq B$; we only note that it is a subtype of $A \to B$ which, when $A = B$, includes the identity function.

▶ **Axiom 1.** *The canonical map* $A =_{\mathcal{U}_i} B \to A \simeq B$ *sending* refl *to* id *is an equivalence.*

We shall also have use for a few other HoTT concepts. First, since univalence causes the *unicity of identity proofs* to fail rather spectacularly, so we isolate those types for which this property still holds as (homotopy) *sets*. We will also use for the stronger properties of being a (homotopy) *proposition*, or *contractible*:

$$\mathsf{isSet}(A) = \textstyle\prod_{a,b:A} \mathsf{isProp}(a = b)$$
$$\mathsf{isProp}(A) = \textstyle\prod_{a,b:A} \mathsf{isContr}(a = b)$$
$$\mathsf{isContr}(A) = \textstyle\sum_{a:A} \prod_{b:A} a = b$$

If $\phi : \mathcal{U} \to \mathcal{U}$ is valued in propositions, we write $\mathcal{U}_\phi$ for $\sum_{A:\mathcal{U}} \phi(A)$. Moreover, we use HProp as condensed notation for $\mathcal{U}_{\mathsf{isProp}}$.

Next, we require a few *higher inductive types* (HITs). These are inductive types that postulate constructors not only of elements of the type, but also of elements of its identity type. For instance, we shall use pushouts $A +_B C$ and localizations as defined by [33]. Since the models of HITs satisfying strict equations is somewhat fraught, we only assume that our HITs satisfy propositional computation rules, i.e., that they are homotopy-initial algebras.

We now turn to the first and most basic axiom of triangulated type theory, which introduces the directed interval.

▶ **Axiom 2.** *There is a set $\mathbb{I}$ equipped with the structure of a bounded distributive lattice $(0, 1, \wedge, \vee)$ such that $0 \neq 1$.*

Crucially, compared with STT we do not assume that $\prod_{i,j:\mathbb{I}} i \leq j \vee j \leq i$ holds. However, just as with the failure of UIP, we may isolate those types that "believe $i \leq j \vee j \leq i$" as *simplicial* types. These are types A such that the following holds:

$$\mathsf{isSimp}(A) = \prod_{i,j:\mathbb{I}} \mathsf{isEquiv}(\mathsf{const} : A \to (i \leq j \vee j \leq i \to A))$$

The next result follows from [33]:

▶ **Proposition 1.** *There is a lex idempotent monad $\boxbslash : \mathcal{U} \to \mathcal{U}$ with $\boxbslash A$ universal among simplicial types receiving a map from A.*

In other words, ordinary simplicial type theory is the "subset of $\mathsf{TT}_{\boxbslash}$" given by types for which $\eta_A : A \to \boxbslash A$ is an equivalence. We write $\mathcal{U}_{\boxbslash} = \sum_{A:\mathcal{U}} \mathsf{isEquiv}(\eta_A)$ for the subtype of $\mathcal{U}$ spanned by simplicial types and note that $\mathcal{U}_{\boxbslash}$ is itself simplicial. Consequently, there is a unique $\hat{\boxbslash} : \boxbslash\mathcal{U} \to \mathcal{U}_{\boxbslash}$ such that $\hat{\boxbslash} \circ \eta = \boxbslash$.

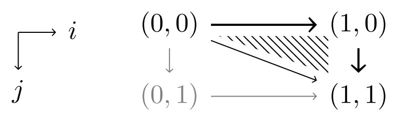
With $\mathbb{I}$, we can make a number of important definitions that isolate those types that actually behave like categories. First, as mentioned earlier, we can define a *synthetic morphism* in a type A to be a map $\mathbb{I} \to A$. Constraining the endpoints of this map, we arrive at the definition of the synthetic hom-type:

$$\hom_A(a, b) = \sum_{f:\mathbb{I}\to A} f\,0 = a \times f\,1 = b$$

Every type comes equipped with a collection of identity morphisms: $\mathsf{id}_a = \lambda\_.\,a : \hom(a, a)$. However, the other operation we expect from a category – composition – is not available by default. What is available instead is a proof-relevant composition *relation*, stating that a pair of synthetic morphisms compose to a third. To define these, we derive the 2-simplex Δ^2 from $\mathbb{I}$. For the sake of future use, we define the n-simplex for any $n \geq 0$:

$$\Delta^n = \sum_{\vec{i}:\mathbb{I}^n} (i_0 \geq i_1 \geq \cdots \geq i_{n-1})$$

Crucially, the 2-simplex can be visualized as a triangle (the shaded region below), if $\mathbb{I}$ is seen as a line segment whence $\mathbb{I}^2$ as a square:



We also have need for the inner horn $\Lambda^2_1 = \sum_{i,j:\mathbb{I}} i = 1 \vee j = 0$, a subtype of Δ^2 which captures the two highlighted segments above.

Say that $\tau : \Delta^2 \to A$ is a witness for the fact that $\tau(-,0), \tau(1,-) : \mathbb{I} \to A$ compose to $\lambda i.\, \tau(i,i)$. A Segal type (sometimes called a pre-category) is one where every pair of composable arrows admits a unique composition witness. To define this formally, we note that maps $\Lambda^2_1 \to A$ precisely capture the data of composable arrows:

▶ **Definition 2.** *A type A is* Segal *if* $\mathsf{isEquiv}(A^{\Delta^2} \to A^{\Lambda^2_1})$ *holds.*

▶ Notation 3. If $f, g : A^{\mathbb{I}}$ and $p : f(1) = g(0)$, we write $[f, g, p]$ for the induced map $A^{\Lambda^2_1}$ and, if A is Segal, $g \circ_p f$ for the composite. Furthermore, we shall subsequently have use for the outer horns $\Lambda^2_0 = \sum_{i,j:\mathbb{I}} i = j \vee j = 0$ and $\Lambda^2_2 = \sum_{i,j:\mathbb{I}} i = 1 \vee i = j$. Finally, we write $\bar{i}$ for the element of Δ^{n+i} given by $(1, \dots 1, 0, \dots)$ of i copies of 1 followed by n copies of 0.

Segal types enjoy a unique composition operation given by the inverse of the map $A^{\Delta^2} \to A^{\Lambda^2_1}$, and calculation shows that the aforementioned definition of the identity morphism is a left and right unit for composition. However, objects in a pre-category have two distinct notions of sameness: via either the identity type *or* synthetic isomorphism. By the latter, we mean a morphism $f : \hom(a, b)$ equipped with $g, h : \hom(b, a)$ along with composition witnesses showing that g (h) is left (right) inverse to f. One can define $\mathbb{E} = \Delta^2 \sqcup_{\Lambda^2_2} \mathbb{I} \sqcup_{\Lambda^2_0} \Delta^2$ such that $\mathbb{E} \to X$ precisely corresponds to an equivalence in X [4, §4.2]). A distinctive feature of ∞-category theory is that these two notions (object equality and isomorphism) can be made to coincide; a property similar to the univalence axiom. We therefore also single out those types which satisfy this local univalence condition:

▶ **Definition 4.** *A type A is* Rezk *if* $\mathsf{isEquiv}(\mathsf{const} : A \to A^{\mathbb{E}})$.

▶ **Definition 5.** *A simplicial, Segal, and Rezk type is called a* category. *A category whose morphisms are all invertible is a* groupoid.

▶ Remark 6. The general results of [33] show that categories and groupoids are modal types for idempotent monads. We write $\bigcirc_{\mathsf{grpd}}$ for the idempotent modality associated with groupoids in particular, i.e., nullification at $\mathbb{I}$.

We shall also have occasion to use the *relative* versions of the Segal and Rezk conditions. Given a family of types $A : X \to \mathcal{U}$, we say that A is (*right*) *orthogonal* to a map $I \to J$ if the following canonical map is an equivalence:

$$\left(\textstyle\sum_{x:X} A(x)\right)^J \to X^J \times_{X^I} \left(\textstyle\sum_{x:X} A(x)\right)^I$$

The relative Segal condition asks that a family of types $A : X \to \mathcal{U}$ be right orthogonal to $\Lambda^2_1 \to \Delta^2$ and the relative Rezk condition asks the same for $\mathbb{E} \to \mathbf{1}$. A Segal family is called *inner* and a family that is both Segal and Rezk is *iso-inner*.

For use in Section 4, we note that we can phrase the requirement that a family be inner using the following predicate:

$$\mathsf{isInner} : (\Delta^2 \to \mathcal{U}) \to \mathsf{HProp}$$
$$\mathsf{isInner}\, A = \mathsf{isEquiv}\left(\left(\textstyle\prod_{t:\Delta^2} A\, t\right) \to \left(\textstyle\prod_{t:\Lambda^2_1} A\, t\right)\right)$$

A family $A : X \to \mathcal{U}$ is inner if and only if $\prod_{h:\Delta^2 \to X} \mathsf{isInner}(A \circ h)$ holds.

▶ Notation 7. We shall often identify a family $A : X \to \mathcal{U}$ with its associated total space projection π from $\widetilde{A} :\equiv \sum_{x:X} A\, x$ to X. We shall say that an arbitrary map of types $f : X \to Y$ is, for instance, inner if the associated map $Y \to \mathcal{U}$ sending y to $f^{-1}(y)$ is inner.

Given a family $A : X \to \mathcal{U}$ and an arrow $f : \mathbb{I} \to X$ we define *dependent arrows* over f from $a : A(f\, 0)$ to $a' : A(f\, 1)$ as follows:

$$\hom^A_f(a, a') :\equiv \textstyle\sum_{\alpha:(t:\mathbb{I}) \to A(f\, t)} (\alpha\, 0 = a) \times (\alpha\, 1 = a')$$

In an inner family there exists an induced composition operation for dependent arrows [4].

2.2 Multimodal type theory

The next step in $\mathsf{TT}_{\boxbslash}$ is to include modalities: special type constructors that violate key properties we ordinarily require in type theory, such as stability under substitution. We use these modalities to internalize crucial operations from our intended model of cubical spaces such as the discrete and codiscrete endofunctors, the opposite functor, etc. To this end, we recall some of the details of the modal extension to type theory, MTT, following [8]. See [7] for a more detailed account. Since our primary goal is to write programs in MTT, we focus on the "informal" version of the syntax.

First, MTT is parameterized by a *mode theory* $\mathcal{M}$. This is a strict 2-category describing the modalities (as 1-cells) and transformations between them (as 2-cells). While MTT also permits distinct type theories to be related by modalities by considering mode theories with multiple modes (0-cells), we do not need this generality and therefore assume that mode theories have only one object. We shall also only be concerned with mode theories with at most one 2-cell between every pair of 1-cells i.e., 2-categories that are merely poset-enriched. As such, our mode theories are simply given by ordered monoids. The mode theory required for $\mathsf{TT}_{\boxbslash}$ is described in Section 2.3, but we continue with an arbitrary mode theory satisfying these constraints for the moment.

The main extension of MTT is to add a new modal type $\langle \mu \mid - \rangle$ for each modality $\mu \in \mathsf{Arr}(\mathcal{M})$. However, as already mentioned modal types are somewhat peculiar, and to accommodate them MTT also modifies context extension. Specifically, each variable in an MTT context is annotated with a "formal division" of modalities $x :_{\mu/\nu} A$. We write Γ/ν for the operation which modifies each annotation in Γ to send $x :_{\mu/\nu_0} A$ to $x :_{\mu/\nu_0 \circ \nu} A$. The variable rule is then modified to account for these formal divisions as follows:

$$\frac{\mu \leq \nu \qquad x :_{\mu/\nu} A \in \Gamma}{\Gamma \vdash x : A}$$

Note that one can recover the ordinary rules for variables by considering the annotation id/id. As a matter of notation therefore, we generally suppress division by id and omit the annotation entirely for id/id so that we instead write $x :_{\mu} A$ or $x : A$.

These annotations are then manipulated by the modal operators $\langle \mu \mid - \rangle$. In particular, they are added by the formation and introduction rules. The (somewhat lengthy) elimination rule, on the other hand, papers over the difference between annotations on a variable and modal types by allowing us to convert a binding $x :_{\nu/\mathsf{id}} \langle \mu \mid A \rangle$ into a binding $y :_{\nu \circ \mu/\mathsf{id}} A$:

$$\frac{\Gamma/\mu \vdash A\ \mathsf{type}}{\Gamma \vdash \langle \mu \mid A \rangle\ \mathsf{type}} \qquad\qquad \frac{\Gamma/\mu \vdash a : A}{\Gamma \vdash \mathsf{mod}_{\mu}(a) : \langle \mu \mid A \rangle}$$

$$\frac{\begin{array}{c}\Gamma/\nu \circ \mu \vdash A\ \mathsf{type} \qquad \Gamma, y :_{\nu/\mathsf{id}} \langle \mu \mid A \rangle \vdash B(y)\ \mathsf{type} \\ \Gamma, x :_{\nu \circ \mu/\mathsf{id}} A \vdash b(x) : B[\mathsf{mod}_{\mu}(x)/y] \qquad \Gamma/\nu \vdash a : \langle \mu \mid A \rangle\end{array}}{\Gamma \vdash \mathsf{let}_{\nu}\ \mathsf{mod}_{\mu}(x) \leftarrow a\ \mathsf{in}\ b(x) : B[a/y]}$$

$$\mathsf{let}_{\nu}\ \mathsf{mod}_{\mu}(x) \leftarrow \mathsf{mod}_{\mu}(a_0)\ \mathsf{in}\ b(x) = b[a_0/x]$$

Aside from these modal types, MTT extends type theory with an extended version of the dependent product type, $(a :_{\mu} A) \to B(a)$. Up to equivalence, these modal dependent products are the same as $(a : \langle \mu \mid A \rangle) \to \mathsf{let}\ \mathsf{mod}_{\mu}(a_0) \leftarrow a\ \mathsf{in}\ B(a_0)$, but offer lighter-weight notation and support a stronger η-rule.

$$\frac{\Gamma/\mu \vdash A\ \mathsf{type} \qquad \Gamma, a :_{\mu} A \vdash b(a) : B(a)}{\Gamma \vdash \lambda a.\, b(a) : (a :_{\mu} A) \to B(a)} \qquad\qquad \frac{\Gamma \vdash f : (a :_{\mu} A) \to B(a) \qquad \Gamma/\mu \vdash a : A}{\Gamma \vdash f(a) : B(a)}$$

When we write "for all $a :_\mu A$, the type $B(a)$ is inhabited" this should be understood to denote a function $(a :_\mu A) \to B(a)$.

Finally, we require the following technical axiom on MTT which governs the interaction between modalities and identity types:

▶ **Axiom 3.** *If $A :_\mu \mathcal{U}$ and $a, b :_\mu A$, then the canonical map $\mathsf{mod}_\mu(a) = \mathsf{mod}_\mu(b) \to \langle \mu \mid a = b \rangle$ sending refl to $\mathsf{mod}_\mu(\mathsf{refl})$ is an equivalence.*

We summarize several basic properties of modalities:

▶ **Proposition 8.**
- *Each $\langle \mu \mid - \rangle$ commutes with $\mathbf{1}$ and pullbacks.*
- *$\langle \mu \mid \langle \nu \mid - \rangle \rangle \simeq \langle \mu \circ \nu \mid - \rangle$ and $\langle \mathsf{id} \mid - \rangle \simeq -$.*
- *If $\mu \leq \nu$, then there is a canonical map $\langle \mu \mid - \rangle \to \langle \nu \mid - \rangle$.*

From the first point, we obtain $\circledast : \langle \mu \mid A \to B \rangle \times \langle \mu \mid A \rangle \to \langle \mu \mid B \rangle$. We often write $f^\dagger$ for $\mathsf{mod}_\mu(f) \circledast -$, when the latter is well-formed.

2.3 Modalities in triangulated type theory

We now turn from MTT generally to the specific instantiation of MTT we require for $\mathsf{TT}_{\boxslash}$. Namely, we will work with the mode theory generated by one 0-cell m, three generating non-identity 1-cells $\{\flat, \sharp, \mathrm{op}\}$, and the following (in)equalities:

$$\flat = \flat \circ \flat = \flat \circ \sharp = \flat \circ \mathrm{op} \qquad \sharp = \sharp \circ \flat = \sharp \circ \sharp = \sharp \circ \mathrm{op} \qquad \flat \leq \mathsf{id} = \mathrm{op} \circ \mathrm{op} \leq \sharp$$

For intuition, in the cubical spaces model of $\mathsf{TT}_{\boxslash}$ the modal operations for the generating modalities are interpreted as follows:
- $[\![\langle \flat \mid - \rangle]\!]$ is the discrete functor – sending a cubical space X to the constant cubical space $\Delta(X([0]))$.
- $[\![\langle \sharp \mid - \rangle]\!]$ is the codiscrete functor – right adjoint to the discrete functor – and sends a cubical space X to the cubical space whose value at $[n]$ is given by $X([0])^{2^n}$.
- $[\![\langle \mathrm{op} \mid - \rangle]\!]$ sends a cubical space X to $X \circ \mathrm{op}$ where op is the functor on the cube category reversing 0 and 1.

If we consider a category $C :_\flat \mathcal{U}$, then these interpretations of $\flat$ and op have very concrete meanings: $\langle \flat \mid C \rangle$ is the underlying discrete groupoid of objects of C and $\langle \mathrm{op} \mid C \rangle$ is the opposite category. In general $\langle \sharp \mid C \rangle$ will fail to be a category even when C is – its utility is mostly in forcing $\flat$ to be a left adjoint.[3] We can prove the following based purely on this mode theory.

▶ **Lemma 9.** *There is an equivalence $\langle \flat \mid A \to \langle \sharp \mid B \rangle \rangle \simeq \langle \flat \mid \langle \flat \mid A \rangle \to B \rangle$ for $A, B :_\flat \mathcal{U}$. Similarly with $\sharp$ and $\flat$ replaced with op.*

Note, however, that the rest of the properties described above for $\flat$ and others are presently just intuitions from a particular model. To actually make these have force within $\mathsf{TT}_{\boxslash}$ itself, we require additional axioms governing the behavior of $\mathbb{I}$ and its interactions with modalities. For reasons of space, we discuss only axioms playing a direct role in the proofs given in the paper. First, we have axioms governing the behavior of $\flat$ and $\mathbb{I}$:

▶ **Axiom 4.** $0, 1 : \mathbb{I}$ *induce an equivalence $\langle \flat \mid \mathsf{Bool} \rangle \simeq \langle \flat \mid \mathbb{I} \rangle$*

[3] We note that the first two modalities are an instance of cohesion [14, 36, 21].

▶ **Axiom 5.** *If $A :_\flat \mathcal{U}$, then* $\mathsf{isEquiv}(\langle\flat \mid A\rangle \to A)$ *if* $\mathsf{isEquiv}(A \to A^{\mathbb{I}})$

Intuitively, the first of this pair states that $\mathbb{I}$ has just two global elements: the specified endpoints 0 and 1. The second axiom ensures that $\mathbb{I}$ "detects" whether a type is of the form $\langle\flat \mid A\rangle$. Phrased pithily: a type A is equivalent to a groupoid of objects if and only if it has no non-invertible synthetic morphisms. These axioms – particularly the last one – are what force $\langle\flat \mid -\rangle$ to match our earlier idea of taking the groupoid core.

The next axiom forces cubes $\mathbb{I}^n$ to form a *separating family* among all types. That is, to detect whether a particular ($\flat$-)map is invertible, it suffices to check whether post-composing with this map induces equivalences between groupoids of maps $\langle\flat \mid \mathbb{I} \to -\rangle$:

▶ **Axiom 6.** *If $A, B :_\flat \mathcal{U}$ and $f :_\flat A \to B$, then f is an equivalence if*

$$\prod_{n:\mathsf{Nat}} \mathsf{isEquiv}((f_*)^\dagger : \langle\flat \mid \mathbb{I}^n \to A\rangle \to \langle\flat \mid \mathbb{I}^n \to B\rangle)$$

Since $\boxslash$ has a universal "mapping-out" property, we have a succinct description of $\boxslash A \to B$ in terms of $A \to B$. In general, there is no such simple relationship between $A \to \boxslash B$ and $A \to B$, with the notable exception of the case when $A = \Delta^n$. Intuitively, $\boxslash B$ ensures that all hyper-cubes in B are determined uniquely by the composite of simplices and so while the hyper-cubes of $\boxslash B$ may be quite different than in B, the collection of *simplices* in B remains unchanged. The following records this relationship:

▶ **Axiom 7.** *For every $A :_\flat \mathcal{U}$, the following holds:*

$$\prod_{n:\mathsf{Nat}} \mathsf{isEquiv}((\eta_*)^\dagger : \langle\flat \mid \Delta^n \to A\rangle \to \langle\flat \mid \Delta^n \to \boxslash A\rangle)$$

Our final axiom in this section records the right adjoint to $\mathbb{I} \to -$ mentioned in Section 1. We must be somewhat more careful in stating this than [16], as we wish to ensure that the postulated adjoint is fully coherent, so we use the following formulation which records that the adjoint exists uniquely:

▶ **Axiom 8.** *For every $A :_\flat \mathcal{U}$, the following holds:*

$$\sum_{A_{\mathbb{I}} :_\flat \mathcal{U}, \epsilon :_\flat (A_{\mathbb{I}})^{\mathbb{I}} \to A} \prod_{B :_\flat \mathcal{U}} \mathsf{isEquiv}(\langle\flat \mid B \to A_{\mathbb{I}}\rangle \to \langle\flat \mid B^{\mathbb{I}} \to A\rangle)$$

Intuitively, $A_{\mathbb{I}}$ is the application of this "amazing" right adjoint to a $\flat$-annotated type A and ϵ is the counit of this adjunction at A. From this data we may reconstruct a functor $(-)_{\mathbb{I}} : \langle\flat \mid \mathcal{U}\rangle \to \langle\flat \mid \mathcal{U}\rangle$. Crucially, this axiom only applies when A is $\flat$-annotated; [16] show that requiring it for arbitrary types forces $\mathbb{I} = \mathbf{1}$, contradicting $0 \neq 1 : \mathbb{I}$. Moreover, as noted earlier, this axiom is *not* validated by the model of STT in simplicial spaces; it is only after shifting to cubical spaces that Axiom 8 is valid. Concretely, it is often the case that even if A is known to be simplicial, the same will not be true of $A_{\mathbb{I}}$. Our main use of this axiom is to "transpose" various $\flat$-annotated predicates $X^{\mathbb{I}} \to \mathsf{HProp}$ into predicates $X \to \mathsf{HProp}$. For convenience, we bundle up this process into the following lemma:

▶ **Lemma 10.** *If $\phi :_\flat \mathcal{U}^{\mathbb{I}^n} \to \mathsf{HProp}$, there is a $\bar{\phi} :_\flat \mathcal{U} \to \mathsf{HProp}$ equipped with a canonical equivalence: $\prod_{A :_\flat X \to \mathcal{U}} \langle\flat \mid (x : X) \to \bar{\phi}(A\,x)\rangle \simeq \langle\flat \mid (x : X^{\mathbb{I}^n}) \to \phi(A \circ x)\rangle$*

Finally, [9] have shown that $\mathsf{TT}_{\boxslash}$ (MTT with this mode theory extended with all of the above axioms) has a model in cubical spaces. They further show, based on a result of [32], that categories in $\mathsf{TT}_{\boxslash}$ are realized by a standard model of ∞-categories: complete Segal spaces [29].

▶ **Theorem 11.** *There is a model of $\mathsf{TT}_{\boxslash}$ in cubical spaces $\mathbf{PSh}_{\mathbf{sSet}}(\square)$. In this model, categories are realized by complete Segal spaces.*

2.4 Category theory in triangulated type theory

We require some of the category theory developed previously in $\mathsf{TT}_{\boxslash}$ and STT [32, 4, 9, 10]. To keep this paper more self-contained, we recall the relevant results and definitions here.

As noted by [32], a natural transformation between functors $f, g : C \to D$ – i.e., an element of $\hom(f, g)$ – corresponds precisely to a family $\prod_{c:C} \hom(f\,c, g\,c)$. Consequently, a pointwise invertible natural transformation is invertible. We note a refinement of this statement by [10] which further reduces this to $\flat$ elements (i.e., objects) of C:[4]

▶ **Lemma 12.** *If $C, D :_\flat \mathcal{U}$ are categories and $f, g :_\flat C \to D$, then a natural transformation $\alpha :_\flat \hom(f, g)$ is invertible if and only if for all $c :_\flat C$ the map $\alpha(c) : \hom(f\,c, g\,c)$ is invertible.*

We similarly have a synthetic version of the classical result that full, faithful and essentially surjective functors are equivalences.

▶ **Lemma 13.** *If $C, D :_\flat \mathcal{U}_{\mathsf{isCat}}$, then $f :_\flat C \to D$ is invertible iff f is essentially surjective and fully faithful on $\flat$-elements of C.*

Our calculations with cocartesian fibrations in Sections 3 and 4 will rely on the theory of adjunctions in $\mathsf{TT}_{\boxslash}$. To begin with, an adjunction between two functors $f : C \to D$ and $g : D \to C$ is given by a collection of equivalences:

$$\alpha : \prod_{c:C} \prod_{d:D} \hom(f\,c, d) \simeq \hom(c, g\,d)$$

Note that we do not require any additional naturality constraints on α, these are automatically enforced by virtue of working synthetically. We say f is a left adjoint if there exists a (necessarily unique) (g, α), and dually that g is a right adjoint if there exists (f, α). It is often difficult to construct such a family of equivalences directly, so we often use the following result of [10]:

▶ **Lemma 14.** *If $C, D :_\flat \mathcal{U}$ are categories, then $f :_\flat C \to D$ is a left adjoint iff for all $d :_\flat D$ there exists $c :_\flat C$ and $\epsilon :_\flat \hom(f(c), d)$ such that the following is an equivalence for all $c' :_\flat C$: $\epsilon_* \circ f : \hom(c', c) \to \hom(f(c'), d)$.*

We shall also have use for the various concrete examples of categories constructed by [9]. Foremost among these is the category of groupoids $\mathcal{S}$ – the ∞-categorical analog of the category of sets. Like our eventual definition of the category of categories, this is characterized through a universal property.

▶ **Definition 15.** *A family of types $A : X \to \mathcal{U}$ is* covariant *if it is right orthogonal to the inclusion $\{0\} \to \mathbb{I}$.*

More intuitively, covariant families are families of groupoids such that synthetic homomorphisms of the base lift coherently to functors of the fibers. We shall give more exposition of this idea indirectly in Section 3 when we study their generalization: cocartesian families. Covariant families are closed under numerous properties, including precomposition, $\sum$-types, etc. We now define $\mathcal{S}$ as the base of the universal covariant family:

[4] [10] prove Lemmas 12 and 14 using the twisted arrow modality, which we have chosen not to include in $\mathsf{TT}_{\boxslash}$ for simplicity. More elementary proofs merely relying on Axiom 6 are possible and so there is no issue with their use in $\mathsf{TT}_{\boxslash}$.

▶ **Definition 16.** *$\mathcal{S}$ is the unique subtype of the universe such that $\mathcal{S} \to \mathcal{U}$ is covariant, and for all $X :_\flat \mathcal{U}$, the canonical map $\langle \flat \mid X \to \mathcal{S} \rangle \to \left\langle \flat \mid \sum_{A:X\to\mathcal{U}} \mathsf{isCov}(A) \right\rangle$ is an equivalence.*

Consequently, the type of objects of $\mathcal{S}$, i.e., $\langle \flat \mid \mathcal{S} \rangle$, is immediately seen to be equivalent to $\flat$-covariant families over $\mathbf{1}$. These, in turn, are equivalent to $\flat$-groupoids. The main result of [9] extends this characterization to synthetic morphisms:

▶ **Theorem 17.** *$\mathcal{S}$ is a category, and there is a canonical equivalence $\hom_{\mathcal{S}}(A, B) \simeq (A \to B)$. Moreover, under this equivalence, identities and composition of synthetic morphisms are realized by identity functions and ordinary function composition.*

Finally, we record a minor but useful result stating that $\mathbb{I}$, and some types derived from it, form categories [9].

▶ **Lemma 18.** *$\mathbb{I}$, $\mathbb{I}^n$, and Δ^n all form categories.*

3 Recollections on cocartesian fibrations

Our eventual goal of characterizing $\langle \flat \mid X \to \mathsf{Cat} \rangle$ crucially depends on the theory of *cocartesian families* [12, 17]. These are a subset of type families $A : X \to \mathcal{U}$ for which (1) each $A\,x$ is a category, and (2) each morphism $f : \hom(x, y)$ in X induces a transport function $A\,x \to A\,y$. We shall require that these transport functions are functorial, and that the coherences enforcing functoriality are themselves coherent, etc. In order to structure all of this data, we ask for A to satisfy a number of propositions that, when combined, give rise to (1) and (2) above. While somewhat indirect, this accounts for the infinite hierarchy of coherences that would otherwise be impossible to write down. This material has been developed within STT [4]. We recall it for the reader's benefit.

3.1 The definition of cocartesian families

Consider a family $A : X \to \mathcal{U}$ and write ρ^A for the canonical map given by restriction and projection $\widetilde{A}^{\Delta^2} \to \widetilde{A}^{\Lambda^2_0} \times_{X^{\Lambda^2_0}} X^{\Delta^2}$. Note that the restriction $\widetilde{A}^{\Delta^2}$ to $\widetilde{A}^{\{\bar{0}\to\bar{1}\}}$ along with the corresponding map $\widetilde{A}^{\Lambda^2_0} \times_{\Lambda^2_0 \to X} X^{\Delta^2} \to \widetilde{A}^{\{\bar{0}\to\bar{1}\}}$ commute with ρ^A, so we may view ρ^A as a map between two families of types over $\mathbb{I} \to \widetilde{A}$. Given $x : \mathbb{I} \to \widetilde{A}$, we denote by ρ^A_x the restriction of ρ^A to the fibers of these families over x.

▶ **Definition 19.** *A morphism $f : \mathbb{I} \to \widetilde{A}$ is* cocartesian *in A (written $\mathsf{isCocartArr}_A(f)$) if ρ^A_f is an equivalence.*

Informally, $f : \mathbb{I} \to \widetilde{A}$ is cocartesian if diagrams of the following shape have a unique lift whenever $\Delta^{\{\bar{0}\to\bar{1}\}} \to \Lambda^2_0 \to \widetilde{A}$ is f:

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d] & \widetilde{A} \arrow[d] \\ \Delta^2 \arrow[r] & X \end{tikzcd}$$

▶ **Definition 20.** *A family $A : X \to \mathcal{U}$ is* cocartesian *if it is iso-inner, each $A(x)$ is simplicial, and the following holds: $\prod_{u:\mathbb{I}\to X} \prod_{a:A(u\,0)} \sum_{f:\hom^A_u(a,\bullet)} \mathsf{isCocartArr}_A(f)$*

This requirement is a proposition [4] and each fiber of such a family is a category, satisfying the first of our requirements.

▶ **Example 21.** For every category A the codomain map $A^{\mathbb{I}} \to A$ is cocartesian. The domain projection is cocartesian iff A has pushouts.

▶ Remark 22. An arrow is *vertical* if it maps to an isomorphism. In a cocartesian fibration, every arrow factors as "vertical ∘ cocartesian".

In case that X is a category a slick characterization of cocartesian families becomes available which implies many closure properties such as under composition, pullback, and Leibniz cotensors [4].

▶ **Proposition 23** ([4])**.** *An iso-inner family $A : X \to \mathcal{U}_{\boxslash}$ over a category X is cocartesian iff the map $(\widetilde{A})^{\mathbb{I}} \to (\widetilde{A})^{\{0\}} \times_{X^{\{0\}}} X^{\mathbb{I}}$ has a left adjoint right inverse.*[5]

For the second desideratum of cocartesian families, we define the cocartesian transport operation, providing the desired functors between fibers. If $A : X \to \mathcal{U}$ is cocartesian and $u : \hom(x, y)$, transport $u_! : A\,x \to A\,y$ is defined by mapping $a : A\,x$ to the codomain of the (unique) cocartesian lift of u starting at a.

▶ **Proposition 24.** *If $A : X \to \mathcal{U}$ is cocartesian, then cocartesian transport is* functorial*, i.e., $(vu)_! = v_! \circ u_!$ and $(\mathsf{id})_! = \mathsf{id}$.*

▶ **Definition 25.** *For $A, B : X \to \mathcal{U}$ cocartesian, we say that $f : \prod_{x:X} A\,x \to B\,x$ is a* cocartesian functor *if f preserves cocartesian arrows. We write $A \to^{\mathrm{cc}} B$ for the type of cocartesian functors.*

For our construction of Cat, it will be helpful to develop the theory of *locally* cocartesian fibrations. These are families $A : X \to \mathcal{U}$ that are cocartesian after restriction $A \circ f$ for every $f : \mathbb{I} \to X$. As setup, for a family $A : \mathbb{I} \to \mathcal{U}$, we call an edge $a : (i : \mathbb{I}) \to A\,i$ *coherently locally cocartesian* if the following proposition holds:

$$\begin{aligned}
&\mathsf{isCohLocallyCoCart} : (A : \mathbb{I} \to \mathcal{U}) \to ((i : \mathbb{I}) \to A\,i) \to \mathsf{HProp} \\
&\mathsf{isCohLocallyCoCart}\,A\,a = \textstyle\prod_{b:(i:\mathbb{I})\to A\,i} \prod_{p:a\,0=b\,0} \\
&\quad \mathsf{isContr}\big(\textstyle\sum_{t:(i,j:\Delta^2)\to A\,i} t|_{\Lambda^2_0} = [a, b, p]\big)
\end{aligned}$$

We then define the structure of having locally cocartesian lifts:

$$\begin{aligned}
&\mathsf{hasCohLCCLifts} : (\mathbb{I} \to \mathcal{U}) \to \mathcal{U} \\
&\mathsf{hasCohLCCLifts}\,A = \textstyle\prod_{a_0:A\,0} \sum_{a:\hom_A(a_0,\bullet)} \mathsf{isLocallyCoCart}\,a
\end{aligned}$$

We extend the preceding definition to general families $A : X \to \mathcal{U}$ by stating that they hold for A if they hold for $A \circ f : \mathbb{I} \to \mathcal{U}$ for all arrows $f : \mathbb{I} \to X$ (or squares $h : \mathbb{I} \times \mathbb{I} \to X$, respectively). We overload the predicates, writing, e.g., $\mathsf{hasCohLCCLifts}(A)$ also for a general family A. A *coherently locally cocartesian family* is one with $\mathsf{hasCohLCCLifts}$ structure.We are indebted to Rob Schellingerhout for explaining the following proof to us [35].

▶ **Theorem 26.** *If $X : \mathcal{U}_{\boxslash}$ and $A : X \to \mathcal{U}_{\boxslash}$ is iso-inner and coherently locally cocartesian, then locally cocartesian edges are cocartesian and A is cocartesian.*

In this case, locally cocartesian lifts are unique, since cocartesian lifts are.

[5] That is, a left adjoint where the unit map is an isomorphism.

3.2 The directed gluing of cocartesian families

We close this section by generalizing the *directed gluing type* inspired by [41] and used in this context by [9]. Roughly, this type takes two cocartesian families over X and a cocartesian functor between them and bundles them into a single cocartesian family over $X \times \mathbb{I}$. This is a key ingredient of our proof of directed univalence, which will eventually amount to a proof that Gl lifts to an equivalence.

Fix cocartesian fibrations $F_0, F_1 : X \to \mathcal{U}_{\boxslash}$ and a cocartesian functor $\alpha : \prod_{x:X} F_0\, x \to F_1\, x$. The *directed gluing* of this data is

$$
\begin{aligned}
&\mathsf{Gl}(F_0, F_1, \alpha) : X \times \mathbb{I} \to \mathcal{U}_{\boxslash} \\
&\mathsf{Gl}(F_0, F_1, \alpha)(x, i) = \textstyle\sum_{f:F_1(x)} i = 0 \to \alpha(x)^{-1}(f)
\end{aligned}
$$

We note that the fibers over $(x, 0)$ and $(x, 1)$ are given by $F_0(x)$ and $F_1(x)$, respectively. Moreover, for each $w : F_0(x)$ there is a map over $\lambda i.\,(x, i)$ connecting w to $\alpha(x, w)$. We show that $\mathsf{Gl}(F_0, F_1, \alpha)$ is iso-inner over $\mathbb{I} \times X$ and that the aforementioned collection of edges make this family cocartesian with transport functor α.

In what follows, we assume that X, F_0, F_1 and α are all $\flat$-annotated. These proofs are all routine applications of orthogonality properties, combined with Lemma 14. We omit the arguments for want of space.

▶ **Lemma 27.** *If X is simplicial, then $\mathsf{Gl}(F_0, F_1, \alpha)$ is iso-inner.*

▶ **Lemma 28.** *If X is a category, then $\mathsf{Gl}(F_0, F_1, \alpha)$ is cocartesian.*

▶ **Corollary 29.** *The transport map $\mathsf{Gl}(F_0, F_1, \alpha)(-, 0) \to \mathsf{Gl}(F_0, F_1, \alpha)(-, 1)$ is given by α.*

▶ **Corollary 30.** *The projection $\pi_0 : \mathsf{Gl}(F_0, F_1, \alpha) \to F_1 \circ \pi_0$ is a cocartesian functor.*

4 The universe of amazingly cocartesian types

We now turn to the construction of Cat. As mentioned in the introduction, Cat will be a subtype of $\mathcal{U}$ and therefore must be classified by a proposition $\mathcal{U} \to \mathsf{HProp}$. The most obvious choice of proposition is something akin to being cocartesian, but a moment's thought reveals this is unworkable: if we are to define a map $\mathsf{isCocartFib} : \mathcal{U} \to \mathsf{HProp}$, what should the input be cocartesian over? Cocartesianness is a property of families!

To fix this, we follow [16] as refined in the context of directed type theories [41, 9]. First, consider a general notion of fibration $\mathsf{isFib}_X : \mathcal{U}^X \to \mathsf{HProp}$. The goal is to define a predicate that witnesses fibrancy of a type A viewed as a family over the entire ambient context. From $\mathsf{isFib}_{\mathbb{I}} :_\flat \mathcal{U}^{\mathbb{I}} \to \mathsf{HProp}$, Lemma 10 yields precisely such a notion of fibrancy. Moreover, this stronger notion of fibration can be shown to agree with the classical notion when we restrict attention to $\flat$-annotated families.

We will now apply this construction, leveraging Theorem 26.

▶ **Lemma 31.** *If $A : X \to \mathcal{U}_{\boxslash}$ is iso-inner, then $\mathsf{hasCohLCCLifts}(A)$ is a proposition.*

Let us write $i : \Delta^2 \to \mathbb{I}^2$ for the canonical inclusion. Using Lemma 10, we now transpose $\mathsf{isInner}(- \circ i)$, and $\mathsf{hasCohLCCLifts}$ to obtain elements of $\mathcal{U} \to \mathcal{U}$, namely $\mathsf{aisInner}$ and $\mathsf{aHasCohLCCLifts}$. Here we have used i as, e.g., $\mathsf{isInner}$ takes $\Delta^2 \to \mathcal{U}$ not $\mathbb{I}^2 \to \mathcal{U}$.

We then define Cat:

$$\mathsf{Cat} := \textstyle\sum_{A:\mathcal{U}_{\boxslash}} \mathsf{isRezk}\, A \times \mathsf{aIsInner}\, A \times \mathsf{aHasLCCLifts}\, A$$

▶ **Lemma 32.** Cat *is a subtype of* $\mathcal{U}_{\boxslash}$.

▶ **Theorem 33.** Cat *is the base of the universal cocartesian family, i.e., for any* $C :_\flat \mathcal{U}_{\boxslash}$*, we have* $\langle \flat \mid C \to \mathsf{Cat} \rangle \simeq \langle \flat \mid \sum_{E:C\to\mathcal{U}} \mathrm{isCocart}(E) \rangle$.

Proof. Fix $A :_\flat X \to \mathcal{U}_{\boxslash}$. Our goal is to show that A factors through Cat if and only if A is cocartesian. To prove this, let us consider the data involved in a factorization through Cat. By definition, this is equivalent to factoring through four distinct subobjects of $\mathcal{U}_{\boxslash}$: those carved out by isRezk, $\mathsf{aIsInner}$, and $\mathsf{aHasCohLCCLifts}$. In the latter two cases, we may analyze these subobjects using the transpositions used to define them.

For instance, if A factors through $\sum_{B:\mathcal{U}_{\boxslash}} \mathsf{aIsInner}(B)$, then there is an element of the following type by Lemma 10: $\langle \flat \mid \prod_{x:X^{\mathbb{I}\times\mathbb{I}}} \mathsf{isInner}(A \circ x \circ i) \rangle$. Such an element exists if and only if A is an inner fibration.

This reasoning applies to $\mathsf{aHasCohLCCLifts}$ so we may conclude that A factors through Cat if and only if it is iso-inner and coherently locally cocartesian. The desired bi-implication is then Theorem 26. ◀

In fact, this same argument shows that the canonical map $\mathsf{Cat} \to \mathcal{U}_{\boxslash}$ is iso-inner; we only require that the base is C is simplicial in the above to apply Theorem 26.

5 The category of categories

In this section, we leverage Theorem 33 to prove the crucial properties of Cat. Namely, we prove Cat is Segal and Rezk, satisfies directed univalence as described in Section 1, and is simplicial. Combining these results together we show that Cat is a category and, in particular, the category of categories.

5.1 Classifying cocartesian fibrations

The main input to the proofs that Cat is Segal and Rezk is a characterization of cocartesian fibrations over $\Delta^n \times C$ where C is a category. To see why, note that by Theorem 33, we know that $f :_\flat X \times \Delta^n \to \mathsf{Cat}$ is determined by a cocartesian family over $X \times \Delta^n$. By giving a precise description of such families, we obtain a more tractable version of, e.g., the restriction map $\langle \flat \mid \mathbb{I}^n \times \Delta^2 \to \mathsf{Cat} \rangle \to \langle \flat \mid \mathbb{I}^n \times \Lambda^2_1 \to \mathsf{Cat} \rangle$. This version will be manifestly invertible, and so we can conclude that Cat is Segal. A key lemma in this process is the following:

▶ **Lemma 34.** *For* $X :_\flat \mathcal{U}$ *a category and* $A, B :_\flat X \to \mathcal{U}$ *cocartesian, a cocartesian functor* $\alpha : \prod_{x:X} A(x) \to B(x)$ *induces an equivalence of total categories* $\widetilde{\alpha} : \widetilde{A} \simeq \widetilde{B}$ *iff* $\prod_{x:_\flat X} \mathsf{isEquiv}(\alpha(x))$ *holds.*

Proof. Since cocartesian families are isofibrations, we know that $\widetilde{A}$ and $\widetilde{B}$ are both categories themselves. By Lemma 13, we check that $\widetilde{\alpha}$ is fully faithful, and essentially surjective.

Essential surjectivity is straightforward: given $(x, b) :_\flat \widetilde{B}$, we take $(x, \alpha(x)^{-1}(b))$ as α is invertible on $\flat$ elements of X. To show that $\widetilde{\alpha}$ is fully faithful, note that transport induces an equivalence between $\mathbb{I} \to \widetilde{A}$ and $\sum_{x:\mathbb{I}\to X} \sum_{a_0:A(x\,0), a_1:A(x\,1)} \hom_{A(x\,1)}(x_! a_0, a_1)$. Since α is a cocartesian functor, we need to only show $\hom_{A(x_1)}(x_! a_0, a_1) = \hom_{B(x_1)}(\alpha(x\,1, x_! a_0), \alpha(x\,1, a_1))$ for $x :_\flat \mathbb{I} \to X$ and $a_\epsilon :_\flat A(x\,\epsilon)$. This holds as $\alpha(x\,1)$ is invertible. ◀

Next we show that every cocartesian family $A :_\flat X \times \mathbb{I} \to \mathcal{U}$, where X is a category, is of the form $\mathsf{Gl}(A(-,0), A(-,1), \lambda x.\,(x,-)_!)$. To this end, we note the following:

▶ **Lemma 35.** *Given A as above, the transport map $\alpha = \lambda x.\,(x,-)_!$ is a cocartesian functor $A(-,0) \to A(-,1)$.*

Proof. Unfolding, this follows from closure under composition and right cancelation which hold for cocartesian arrows [4, Proposition 5.1.8]. ◀

Consequently, $B = \mathsf{Gl}(A(-,0), A(-,1), \alpha)$ is a cocartesian family. Moreover, we can produce a map of families $A \to B$:

$$\mathsf{glue}(x,i,a) = (x,i,((x,i \vee -)_! a, \lambda p : i = 0.\, p_! a))$$

In words, we use cocartesian transport to move $a : A(x,i)$ to $A(x,1)$ and, if $i = 0$ to begin with, record the original a as well.

▶ **Lemma 36.** $\mathsf{glue} : \prod_{p:X\times\mathbb{I}} A(p) \to B(p)$ *is a cocartesian functor.*

▶ **Corollary 37.** glue *is an equivalence of cocartesian families.*

Proof. Applying Lemma 34, it suffices to check this equivalence fiberwise on $\flat$-annotated elements $(x,i) :_\flat X \times \mathbb{I}$. In particular, it suffices to check that induces an equivalence on $\flat$-annotated elements of $\mathbb{I}$ which, by Axiom 4, consists only of 0 and 1. However, over 0 and 1 we see that glue is an equivalence: over 0, this is immediate and over 1 it follows from the observation that cocartesian transport over the identity arrow is the identity. ◀

Thus, a cocartesian family $A :_\flat X \times \mathbb{I} \to \mathcal{U}$ is fully described by its restrictions to 0 and 1 along with the associated transport map. Combining this classification result in the case where $X = \mathbf{1}$ with Theorem 33, we obtain the directed univalence principle:

▶ **Corollary 38** (Directed univalence)**.** *If A, B are $\flat$-elements of Cat, then there is an equivalence* $\mathsf{dua} : \hom_{\mathsf{Cat}}(A,B) \simeq \langle \flat \mid A \to B \rangle$.

We note that, in general, we actually obtain an equivalence between $\langle \flat \mid X \times \mathbb{I} \to \mathsf{Cat} \rangle$ and $\left\langle \flat \mid \sum_{A_0,A_1:\mathsf{Cat}^X} A_0 \to^{\mathrm{cc}} A_1 \right\rangle$. For what follows, we also require a similar accounting of cocartesian families $A :_\flat X \times \Delta^2 \to \mathcal{U}$. The story plays out in much the same way; we define B_1 as the following iterated glue type:

$$B_0 = \mathsf{Gl}(A(-,\bar{0}), A(-,\bar{1}), \lambda x.\,(x,-)_!) : X \times \mathbb{I} \to \mathcal{U}_{\boxslash}$$
$$B_1 = \mathsf{Gl}(B_0, A(-,\bar{2}) \circ \pi_0, \lambda (x,i)\,(a_1,\_).\,(x, \bar{1} \vee - \wedge \bar{2})_! a_1) : (X \times \mathbb{I}) \times \mathbb{I} \to \mathcal{U}_{\boxslash}$$

Combining Corollary 30 and Lemma 35 with Lemma 28, we conclude that B_1 is a cocartesian family $X \times \mathbb{I}^2 \to \mathcal{U}$. We then take B to be the restriction of B_1 to $X \times \Delta^2$. The map of families glue considered previously easily generalizes to this Δ^2 case and we may prove the following:

▶ **Lemma 39.** *The map* $\mathsf{glue}_2 : \prod_{p:X\times\Delta^2} A\,p \to B\,p$ *is cocartesian.*

Again, an application of Lemma 34 allows us to conclude that glue_2 is an equivalence. Combining these steps with Theorem 33 once more, we conclude the following:

▶ **Corollary 40.** *Cocartesian transport induces an equivalence* $\langle \flat \mid \mathsf{Cat}^{X\times\Delta^2} \rangle \simeq \left\langle \flat \mid \sum_{A_0,A_1,A_2:\mathsf{Cat}^X} A_0 \to^{\mathrm{cc}} A_1 \times A_1 \to^{\mathrm{cc}} A_2 \right\rangle$.

5.2 Cat is Segal and Rezk

Having characterized both cocartesian fibrations over $X \times \Delta^2$ and $X \times \mathbb{I}$ for all categories X, it is only slightly more work to prove that Cat is both Segal and Rezk.

▶ **Lemma 41.** Cat *is Segal.*

Proof. We wish to show that $\mathsf{Cat}^{\Delta^2} \to \mathsf{Cat}^{\Lambda^2_1}$ is an equivalence. Using Axiom 6, it suffices to show that the following map is an equivalence for all n: $\langle \flat \mid \mathbb{I}^n \times \Delta^2 \to \mathsf{Cat} \rangle \to \langle \flat \mid \mathbb{I}^n \times \Lambda^2_1 \to \mathsf{Cat} \rangle$. Note that $\langle \flat \mid \mathbb{I}^n \times \Lambda^2_1 \to \mathsf{Cat} \rangle$ is equivalent to the following:

$$\langle \flat \mid \mathbb{I}^n \times \mathbb{I} \to \mathsf{Cat} \rangle \times_{\langle \flat \mid \mathbb{I}^n \to \mathsf{Cat} \rangle} \langle \flat \mid \mathbb{I}^n \times \mathbb{I} \to \mathsf{Cat} \rangle$$

This follows from the fact that $\langle \flat \mid - \rangle$ preserves pullbacks together with the fact that $\Lambda^2_1 = \mathbb{I} \sqcup_{\mathbf{1}} \mathbb{I}$. Next, since $\mathbb{I}^n$ is a category, the results of the previous section allow us to rephrase the above map into the following:

$$\left\langle \flat \mid \textstyle\sum_{A,B,C:\mathsf{Cat}^{\mathbb{I}^n}} A \to^{\mathrm{cc}} B \times B \to^{\mathrm{cc}} C \right\rangle \to$$
$$\left\langle \flat \mid \textstyle\sum_{A,B:\mathsf{Cat}^{\mathbb{I}^n}} A \to^{\mathrm{cc}} B \right\rangle \times_{\langle \flat \mid \mathsf{Cat}^{\mathbb{I}^n} \rangle} \left\langle \flat \mid \textstyle\sum_{B,C:\mathsf{Cat}^{\mathbb{I}^n}} B \to^{\mathrm{cc}} C \right\rangle$$

This being an equivalence follows immediately from the fact that $\langle \flat \mid - \rangle$ preserves pullbacks by virtue of Axiom 3. ◀

Inspecting this proof, we see that if $P : \Lambda^2_1 \to \mathsf{Cat}$, the resulting composed edge $f : \mathbb{I} \to \mathsf{Cat}$ is a cocartesian fibration where $f(0) = P(\bar{0})$, $f(1) = P(\bar{2})$ and cocartesian transport from 0 to 1 is the composite of transporting in P from $\bar{0}$ to $\bar{1}$ and then from $\bar{1}$ to $\bar{2}$.

▶ **Lemma 42.** Cat *is Rezk.*

Proof. We wish to show that all synthetic isomophisms in Cat are equivalent to the identity. Once more, we apply Axiom 6 showing that the restriction map $\langle \flat \mid \mathbb{I}^n \to \mathsf{Cat} \rangle \to \langle \flat \mid \mathbb{I}^n \times \mathbb{E} \to \mathsf{Cat} \rangle$ is an equivalence. Commuting $\langle \flat \mid - \rangle$ with the pullback arising from mapping out of $\mathbb{E}$ once more, we may instead consider the map:

$$\langle \flat \mid \mathbb{I}^n \to \mathsf{Cat} \rangle \to \langle \flat \mid \mathsf{Cat}^{\mathbb{I}^n \times \Delta^2} \rangle \times_{\dots} \langle \flat \mid \mathsf{Cat}^{\mathbb{I}^n \times \mathbb{I}} \rangle \times_{\dots} \langle \flat \mid \mathsf{Cat}^{\mathbb{I}^n \times \Delta^2} \rangle$$

Applying the results of the previous section and commuting $\langle \flat \mid - \rangle$ past pullbacks, we may recast the above:

$$\langle \flat \mid \mathbb{I}^n \to \mathsf{Cat} \rangle \to \left\langle \flat \mid \textstyle\sum_{A,B:\mathsf{Cat}^{\mathbb{I}^n}} \sum_{f:A \to^{\mathrm{cc}} B} \sum_{g,h:B \to^{\mathrm{cc}} A} f \circ g = \mathsf{id} \times h \circ f = \mathsf{id} \right\rangle$$

However, since Cat is a subtype of $\mathcal{U}$ it satisfies the univalence axiom. The result then follows as the data imposed on f ensures that it is a family of equivalences. ◀

5.3 Cat is simplicial

Our remaining task is to show that Cat is simplicial. Unfortunately, this is far from immediately obvious. After all, Cat is defined using the amazing right adjoint to $\mathbb{I} \to -$, which is the primary source of non-simplicial types. Fortunately, in this case we may use Theorem 33 to produce a left section to $\eta : \mathsf{Cat} \to ⧅\mathsf{Cat}$ and this implies that η is an equivalence [33, Lemma 1.20]. To begin with, we record the following lemma:

▶ **Lemma 43.** *The commuting square between* $\mathsf{Cat}_\bullet = \left(\sum_{A:\mathsf{Cat}} A\right) \to \mathsf{Cat}$ *and* $⧅\mathsf{Cat}_\bullet \to ⧅\mathsf{Cat}$ *induced by* η *is cartesian.*

Proof. Unfolding definitions and comparing fibers, this follows from the fact that each $A : \mathsf{Cat}$ is simplicial and $\boxbslash$ is lex. ◀

Next, we note that if $\boxbslash\mathsf{Cat}_\bullet \to \boxbslash\mathsf{Cat}$ represents a cocartesian family itself, then there must be a unique classifying map $\boxbslash\mathsf{Cat} \to \mathsf{Cat}$ and, pasting together the relevant pullback squares, we obtain the following composite pullback diagram:

$$\begin{tikzcd}
\mathsf{Cat}_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \boxbslash\mathsf{Cat}_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cat}_\bullet \arrow[d] \\
\mathsf{Cat} \arrow[r] & \boxbslash\mathsf{Cat} \arrow[r] & \mathsf{Cat}
\end{tikzcd}$$

By the univalence property of Cat, this bottom composite must be the identity. Consequently, the classifying map $\boxbslash\mathsf{Cat} \to \mathsf{Cat}$ is the required left inverse to the unit. All that remains, therefore, is to prove that $\boxbslash\mathsf{Cat}_\bullet \to \boxbslash\mathsf{Cat}$ is a cocartesian fibration.

To prove this we will use Axiom 7 along with Lemma 14 and Proposition 23. First, we must note that $\boxbslash\mathsf{Cat}$ and $\boxbslash\mathsf{Cat}_\bullet$ are both categories. We therefore record the following result:

▶ **Lemma 44.** *If $X :_\flat \mathcal{U}$ is Segal and Rezk, then $\boxbslash X$ is a category.*

▶ **Lemma 45.** *The family $\boxbslash\mathsf{Cat}_\bullet \to \boxbslash\mathsf{Cat}$ is cocartesian.*

Proof. We wish to show that the canonical map $\boxbslash\mathsf{Cat} \to \mathcal{U}_{\boxbslash}$ factors through the subtype Cat. By Axiom 6, we may test this after restricting $\boxbslash\mathsf{Cat} \to \mathcal{U}_{\boxbslash}$ along some map $x :_\flat \mathbb{I}^n \to \boxbslash\mathsf{Cat}$. Furthermore, by Axiom 7, if we further restrict along any map $x' :_\flat \Delta^m \to \mathbb{I}^n$, the composite $x \circ x'$ factors through Cat and the resulting family over Δ^m is therefore cocartesian.

Our goal is to show that $E = (\boxbslash\mathsf{Cat}_\bullet \times_{\boxbslash\mathsf{Cat}} \mathbb{I}^n) \to \mathbb{I}^n$ is cocartesian. We first observe that it is iso-inner (as a map between categories). Furthermore, we note that every $\flat$-annotated map $f :_\flat \mathbb{I} \to \mathbb{I}^n$ and element $e :_\flat E_{f\,0}$ admits a cocartesian lift, following our assumption about the restriction of $E \to \mathbb{I}^n$ to simplices.

Finally, to show that this family is cocartesian, we wish to show that $E^{\mathbb{I}} \to (\mathbb{I}^n)^{\mathbb{I}} \times_{\mathbb{I}^n} E$ has a left adjoint right inverse. By Lemma 14 it suffices to check that this property holds when restricted to $\flat$-annotated elements of $(f, e_0) : (\mathbb{I}^n)^{\mathbb{I}} \times_{\mathbb{I}^n} E$. We have already observed that there exist cocartesian lifts over such elements f and e_0, and by [4], this is precisely what is required to construct the relevant left adjoint at f, e_0. ◀

▶ **Corollary 46.** Cat *is a category.*

6 Full straightening-unstraightening

In this section, we prove Lurie's straightening–unstraightening theorem which states that for a category $C :_\flat \mathcal{U}$, the type $C \to \mathsf{Cat}$ is equivalent to the subcategory of $\mathsf{Cat}_{/C}$ (that is, $\sum_{f : \mathsf{Cat}^{\mathbb{I}}} f(1) = C$) restricted such that its 0- and 1-cells are given cocartesian families and cocartesian functors. Accordingly, this section let us fix $C :_\flat \mathcal{U}$ a category.

We will construct a map $U : (C \to \mathsf{Cat}) \to \mathsf{Cat}_{/C}$ and prove that it is (1) an embedding such that (2) its image on $\flat$-annotated elements $C \to \mathsf{Cat}$, and $\mathbb{I} \to (C \to \mathsf{Cat})$ satisfies precisely the above criteria. From this, we show that $(C \to \mathsf{Cat}) \to \mathsf{Cat}_{/C}$ satisfies the universal property of the subcategory $\mathsf{Cocart}(C)$ of *cocartesian families over C* (Corollary 52).

▶ Remark 47. The material in this section follows [5] with minor alterations to make it more convenient in $\mathsf{TT}_{\boxslash}$. In particular, it is from there that we learned of this construction the unstraightening functor and characterization of its image. That such an adaptation is possible is expected but encouraging: the axiomatic approach given by op. cit. is intended to give high-level arguments which can be translated into formal systems satisfying their axioms and our construction of Cat ensures that $\mathsf{TT}_{\boxslash}$ proves all the axioms used in this argument.

▶ Remark 48. We avoid explicitly constructing $\mathsf{Cocart}(C)$ to avoid the detour of describing the construction of non-full subcategories. Such constructions are possible using e.g., $(-)_{\mathbb{I}}$.

6.1 The unstraightening map

We begin by constructing a map U from $C \to \mathsf{Cat}$ to $\mathsf{Cat}_{/C}$. We break this process into several steps. We begin by considering two particular cocartesian families over $C \to \mathsf{Cat}$:

$$E(f) = \textstyle\sum_{c:C} f(c) \qquad B(f) = C$$

These are both cocartesian families over C and the canonical projection is a cocartesian map $\pi_0 : E \to^{\mathrm{cc}} B$ as well – and therefore a cocartesian functor. We may therefore glue these together to form a cocartesian family: $\mathsf{Gl}(E, B, \pi_0) : (\mathsf{Cat}^C) \times \mathbb{I} \to \mathsf{Cat}$. First, let us compute $\mathsf{Gl}(E, B, \pi_0)(-, 1)$ and observe that it is canonically identified to $\lambda\_. C$, via the following family of paths:

$$\Phi = \lambda f.\, \mathsf{ua}(\pi_0) : \textstyle\prod_{f:C\to\mathsf{Cat}} \mathsf{Gl}(E, B, \pi_0)(f, 1) = C$$

By transposing and using this identification, we obtain:

$$U : (C \to \mathsf{Cat}) \to \mathsf{Cat}_{/C} \qquad U = \lambda f.\, (\mathsf{Gl}(E, B, \pi_0)(f, -), \Phi(f))$$

6.2 The image of the unstraightening map

Our next task to identify the image of U and, in particular, to show that it is precisely the category of cocartesian families over C and cocartesian functors between them. We therefore compute fibers of $\langle \flat \mid \mathsf{Cat}^C \rangle \to \langle \flat \mid \mathsf{Cat}_{/C} \rangle$ and $\langle \flat \mid \mathbb{I} \to \mathsf{Cat}^C \rangle \to \langle \flat \mid \mathbb{I} \to \mathsf{Cat}_{/C} \rangle$.

In particular, we will show that the fiber over $p :_\flat \mathsf{Cat}_{/C}$ is precisely the proposition stating whether p is cocartesian and over $f :_\flat \mathbb{I} \to \mathsf{Cat}_{/C}$ it corresponds to the triple of propositions requiring $f(0)$ and $f(1)$ to be cocartesian and f itself to induce a map of cocartesian families. In light of the Segal condition, this characters the fibers over arbitrary simplices and, via Axiom 6 and the simpliciality of Cat, proves that U is an embedding. Moreover, our description of the fibers shows that the resultant subcategory of $\mathsf{Cat}_{/C}$ is precisely as described at the beginning of this section.

▶ **Lemma 49.** *The fiber of U over $p :_\flat \mathsf{Cat}_{/C}$ is a proposition inhabited iff p is cocartesian.*

Proof. Post-composing with directed univalence, we may identify a fiber of U with a fiber of the map $U' : \langle \flat \mid C \to \mathsf{Cat} \rangle \to \left\langle \flat \mid \sum_{E:\mathsf{Cat}} E \to C \right\rangle$. Consider a category $E :_\flat \mathsf{Cat}$ and a function $\pi_E :_\flat E \to C$. Our goal is to compute the fiber $\sum_{f:\langle \flat \mid C \to \mathsf{Cat}\rangle} U'(f) = \mathsf{mod}_\flat(E, \pi_E)$.

Unfolding U', an element $f :_\flat C \to \mathsf{Cat}$ is sent to (E, π_E) if and only if we have an equivalence $e :_\flat E \simeq \sum_{c:C} f(c)$ and an equation $p :_\flat \pi_E = \pi_0 \circ e$. By another application of univalence, this is equivalent to requiring that $\langle \flat \mid \pi_E^{-1}(-) = f \rangle$, so the fiber amounts to $\sum_{f:_\flat C\to\mathsf{Cat}} \langle \flat \mid \pi_E^{-1} = f \rangle$. This is a proposition by Proposition 8 and by Theorem 33 inhabited iff π_E is a cocartesian family. ◀

▶ **Lemma 50.** *The fiber of U over $f :_\flat \mathbb{I} \to \mathsf{Cat}_{/C}$ is a proposition inhabited iff f is a cocartesian functor between cocartesian families.*

Proof. Rearranging equations, we may identify $\langle \flat \mid \mathbb{I} \to \mathsf{Cat}_{/C} \rangle$ with $\big\langle \flat \mid \sum_{h:\Delta^2 \to \mathsf{Cat}} h(\bar{2}) = C \big\rangle$ which we may then identify via directed univalence with $\big\langle \flat \mid \sum_{E,F:\mathsf{Cat}} E \to F \times F \to C \big\rangle$. Post-composing U with these maps, we instead consider the following:

$$U' : \langle \flat \mid \mathbb{I} \to \mathsf{Cat}^C \rangle \to \big\langle \flat \mid \textstyle\sum_{E,F:\mathsf{Cat}} E \to F \times F \to C \big\rangle$$

Consider categories $E, F :_\flat \mathsf{Cat}$ and functions $\pi_F :_\flat F \to C$ and $\alpha :_\flat E \to F$. Our goal is to compute the fiber of U' over this data. Unfolding, an element $h :_\flat \mathbb{I} \to \mathsf{Cat}^C$ is sent to E, F, π_F if and only if we have the following:

- an equivalence $e_0 :_\flat E \simeq \sum_{c:C} h(0, c)$,
- an equivalence $e_1 :_\flat F \simeq \sum_{c:C} h(1, c)$,
- a path $\phi :_\flat e_1 \circ \alpha = \beta \circ e_0$ where $\beta = \lambda c.\, (-, c)_!$ is given by the cocartesian transport of h
- a path $\phi :_\flat \pi_F = \pi_0 \circ e_1$.

Once more using univalence, we are reduced to $\flat$-annotated equation between maps of families over C (i.e. $C \to \sum_{A,B:\mathcal{U}} B^A$):

$$\lambda c.\, \pi_0 : \textstyle\prod_{c:C} \big(\sum_{f:\pi_F^{-1}(c)} \alpha^{-1}(f)\big) \to \pi_F^{-1}(c) \qquad \lambda c.\, (-, c)_! : \prod_{c:C} h(0, c) \to h(1, c)$$

Since $\langle \flat \mid \mathbb{I} \to \mathsf{Cat}^C \rangle$ embeds into $\big\langle \flat \mid C \to \sum_{A,B:\mathcal{U}} B^A \big\rangle$ by directed univalence, Proposition 8 ensures this fiber over (E, F, π_F) is a proposition. The same analysis shows that it is inhabited iff $\pi_F \circ \alpha$ and π_E are cocartesian and α is a map of cocartesian families. ◀

▶ **Corollary 51.** *The map $U : (C \to \mathsf{Cat}) \to \mathsf{Cat}_{/C}$ is an embedding.*

In fact, in light of our identification of the fibers of U we may also characterize when a functor lifts along it. This is the universal property of $\mathsf{Cocart}(C)$ mentioned earlier:

▶ **Corollary 52** (Straightening–unstraightening)**.** *If $D :_\flat \mathcal{U}$ is a category, a map $f :_\flat D \to \mathsf{Cat}_{/C}$ lifts along U to Cat^C if and only if*

1. *for each $d :_\flat D$, the functor $f(d)$ is a cocartesian family.*
2. *for each $d :_\flat \mathbb{I} \to D$, the functor induced by $f \circ d : \mathbb{I} \to \mathsf{Cat}_{/C}$ is a cocartesian functor between the cocartesian families.*

7 Examples

We have thus far focused on the construction of Cat and verifying its essential properties and so we close by discussing some of the new examples and category theory unlocked by Cat. For reasons of space, we content ourselves with only sketching several examples.

7.1 Subcategories of Cat

We begin by noting that since every covariant family is cocartesian, there is a unique map from the base of the universal covariant family $\mathcal{S}$ to the base of the universal cocartesian family Cat. This is the inclusion of groupoids into categories.

▶ **Lemma 53.** *The map $i : \mathcal{S} \to \mathsf{Cat}$ is fully faithful and possesses both left and right adjoints: $|-| \dashv i \dashv (-)^{\simeq}$.*

Proof Sketch. The second half of this statements follows from Lemma 14. In particular, we use this lemma to extend the pointwise assignments of $X :_\flat \mathsf{Cat} \mapsto \langle \flat \mid X \rangle : \mathcal{S}$ and $X :_\flat \mathsf{Cat} \mapsto \bigcirc_{\mathsf{grpd}} X : \mathcal{S}$ to functors $\mathsf{Cat} \to \mathcal{S}$. The fact that i is fully faithful is then immediate from Axiom 5: if X is a groupoid then the unit $X \to \langle \flat \mid i(X) \rangle$ is an equivalence. A standard argument shows that η being invertible implies the left adjoint is fully faithful. ◀

Many other interesting categories exist as full subcategories of Cat. For instance, we may isolate univalent 1-categories as the full subcategory of Cat [9, §7] given by the following predicate:

$$\mathsf{is1Cat} : (\flat \mid \mathsf{Cat}) \to \mathsf{HProp}$$
$$\mathsf{is1Cat}(C) = \textstyle\prod_{a,b :_\flat C} \mathsf{isSet}(\hom_C(a,b))$$

Similar definitions immediately yield $(n,1)$-categories for all n. Notably, by restricting to $n = -1$ we obtain the category of partial orders and, restricting further to linear partial orders, the simplex category $\Delta \hookrightarrow \mathsf{Cat}$. In fact, the same argument as was used to $\mathcal{S} \hookrightarrow \mathsf{Cat}$ allows us to prove the following:

▶ **Lemma 54.** *The inclusion of $\mathsf{Cat}_n \hookrightarrow \mathsf{Cat}$ is a right adjoint.*

Proof Sketch. One adapts Lemma 53 to use the modality nullifying the maps $\Lambda^2_1 \to \Delta^2$, $\mathbb{E} \to \mathbf{1}$, and $\partial\Delta^{n+2} \to \Delta^{n+2}$. ◀

For a small example of how these ingredients might be combined to build a useful and important construction in higher category theory, we turn our attention to monoidal categories and their algebraic K theory.

Let us write $[n]$ for the element of Δ realizing the linear order $\{0 \leq \cdots \leq n\}$. Using Corollary 38, we define $\rho^i_n : \hom([1],[n])$ which sends $0 \leq 1$ to $\{i \leq i+1\} \subseteq [n]$.

▶ **Definition 55.** *A monoidal category $C^\otimes : \mathsf{Cat}^{\langle \mathrm{op} \mid \Delta \rangle}$ is a functor where $(\rho^1_n, \ldots, \rho^n_n) : C^\otimes([n]) \to C^\otimes([1]) \times \cdots \times C^\otimes([1])$ is an equivalence for all n.*

Replacing Cat by $\mathcal{S}$ in the above gives the definition of an E_1-monoid: a homotopy-coherent monoid [9, §7].

▶ **Definition 56.** *The category of monoidal categories MCat is the full subcategory of $\langle \mathrm{op} \mid \Delta \rangle \to \mathsf{Cat}$ spanned by monoidal categories.*

We readily adapt this definition to (1) the category of E_1-monoids Mon as a subcategory of $\mathcal{S}^{\langle \mathrm{op} \mid \Delta \rangle}$ and to (2) the category of monoidal 1-categories MCat_1 as a subcategory of $\mathsf{Cat}_1^{\langle \mathrm{op} \mid \Delta \rangle}$.

As both $(-)^\simeq : \mathsf{Cat} \to \mathcal{S}$ and the inclusion $\mathsf{Cat}_1 \to \mathsf{Cat}$ are right adjoints, they preserve finite products and therefore post-composing by these maps induces functors $\mathsf{MCat} \to \mathsf{Mon}$ and $\mathsf{MCat}_1 \to \mathsf{MCat}$. We note next that – viewing Mon as a subcategory of $\mathcal{S}^{\langle \mathrm{op} \mid \Delta \rangle}$ – we may take the colimit of $M : \mathsf{Mon}$ to obtain a space $\varinjlim M$. In fact, this space is canonically pointed: the initial object in Mon is the functor $\mathsf{const}\,\mathbf{1}$ and $\varinjlim \mathsf{const}\,\mathbf{1} = \mathbf{1}$. Finally, regarding the loop-space functor as a map $\Omega : \mathcal{S}_* \to \mathcal{S}_*$ we define k to be the following chain of functors:

$$k : \mathsf{MCat}_1 \to \mathsf{MCat} \to \mathsf{Mon} \to \mathcal{S}_* \to \mathcal{S}_*$$

We may now define the simplest form of algebraic K-theory:

▶ **Definition 57** (Quillen)**.** *The ith* K-group *of a monoidal 1-category $C^\otimes$ is the ith homotopy group $K_i(C^\otimes) = \pi_i(k(C^\otimes))$.*

Notably, with Cat to hand all of these definitions are quite conceptual and automatically functorial. We emphasize that this is only a first step towards realizing K-theory. We leave it to future work to show e.g., that modules over a ring are a monoidal category.

7.2 The structure homomorphism principle

To give a different class of examples involving Cat, we turn to the *structure homomorphism principle*. This is the directed enhancement of HoTT's structure identity principle. This principle states that by taking ordinary type-theoretic definitions of objects in a certain category but using Cat or $\mathcal{S}$ instead of $\mathcal{U}$, we obtain the correct synthetic category with the expected homomorphisms.

For a simplest example of this phenomena:

▶ **Lemma 58.** *The type $\sum_{A:\mathsf{Cat}} A$ is the* lax *slice $\mathbf{1} /\!\!/ \mathsf{Cat}$ i.e. its objects are pointed categories (C,c) and when $(C,c),(D,d) :_\flat \sum_{A:\mathsf{Cat}} A$ morphisms $\hom((C,c),(D,d))$ consist of functions $f : C \to D$ together with a morphism $\hom(f(c),d)$.*

Proof. As $\sum_{A:\mathsf{Cat}} A$ is the total space of the cocartesian family $\mathsf{Cat} \to \mathcal{U}$, it is a category. The characterization of objects is immediate from Proposition 8. For morphisms, we use Corollary 38 with the factorization of a morphism in a total space of a cocartesian family into a cocartesian map followed by a vertical map. ◀

More generally, one may show that $\sum_{A:\mathsf{Cat}} A^C$ is the lax slice category $C /\!\!/ \mathsf{Cat}$ for any $C :_\flat \mathsf{Cat}$. For a more sophisticated example, consider the following type built using the right adjoint to $i : \mathcal{S} \to \mathsf{Cat}$.

$$\mathsf{Cat}_{\text{smarked}} = \textstyle\sum_{C:\mathsf{Cat}}((\mathbb{I} \to C)^{\simeq} \to \mathsf{Bool})$$

By directed univalence applied to $\mathcal{S}$, this type is equivalent to $\sum_{C:\mathsf{Cat}} \hom((\mathbb{I} \to C)^{\simeq}, \mathsf{Bool})$ which is easily seen to be a category. Objects of $\mathsf{Cat}_{\text{smarked}}$ are a pairs of (1) a category $C :_\flat \mathsf{Cat}$ and (2) a (decidable) predicate ϕ_C on the groupoid $\langle \flat \mid \mathbb{I} \to C \rangle$ (recall that $(-)^{\simeq}$ extends $\langle \flat \mid - \rangle$). The morphisms in $\mathsf{Cat}_{\text{smarked}}$ are then functors between the underlying category $f : C \to D$ such that $\phi_C = \phi_D \circ f$. This is almost the category of *marked* categories (the category of categories with a distinguished class of morphisms along with functors preserving these morphisms) but the morphisms are off: we should expect only that ϕ_C implies $\phi_D \circ f$.

To rectify this, we should replace Bool with the non-discrete category $\mathbb{I}$. Just as with $\sum_{A:\mathsf{Cat}} A$, this introduces the required laxity in the morphism. Unfortunately, however, we can no longer rely on directed univalence for $\mathcal{S}$ in this case; we must use Corollary 38 which only applies to $\flat$-annotated elements. This makes it much more difficult to prove that $\sum_{C:\mathsf{Cat}}((\mathbb{I} \to C)^{\simeq} \to \mathbb{I})$ is a category. Systematically handling these "mixed-variance" applications of SHP requires more exploration of exponentiable functors [2]: we must show that (co)cartesian functors are exponentiable. We defer this to future work and instead work relative to the following conjecture:

▶ **Conjecture 59.** *If $C :_\flat \mathsf{Cat}$ then $- \to C : \mathcal{S} \to \mathsf{Cat}$ is cartesian where the cartesian lift of $f :_\flat \hom_{\mathcal{S}}(A,B)$ to (B,ϕ) is $(A, \phi \circ f)$.*

The difficult piece of this conjecture is establishing that $- \to C$ is an iso-inner family, with the cartesian lifts following directly from the dual of Proposition 23. To see this enables further applications of SHP, note that choosing $C = \mathbb{I}$ yields:

▶ **Lemma 60.** *The category $\mathsf{Cat}_{\text{marked}} = \sum_{C:\mathsf{Cat}}((\mathbb{I} \to C)^{\simeq} \to \mathbb{I})$ is the category of marked categories.*

Proof. We discuss only the characterization of synthetic morphisms. First, note that $\pi_0 : \left(\sum_{C:\mathsf{Cat}}((\mathbb{I} \to C)^{\simeq} \to \mathbb{I})\right) \to \mathsf{Cat}$ is a pullback of the family described in Conjecture 59 and therefore cartesian. We may then factor any morphism $f :_\flat \hom_{\mathsf{Cat}_{\mathrm{marked}}}((C, \phi_C), (D, \phi_D))$ uniquely into a cartesian lift of a map $f_0 : C \to D$ along with $f_1 : \hom_{\mathbb{I}}(\phi_C, \phi_D \circ ((f_0)_*)^{\simeq})$. This is precisely the data of a morphism of marked categories. ◀

8 Conclusions and related work

In this work, we have constructed a subtype of the universe $\mathsf{Cat} \hookrightarrow \mathcal{U}$ and shown that it gives rise to the *category of categories* within STT. In particular, we have shown that it is simplicial, Segal, and Rezk and characterized its objects and mapping spaces to be categories and functors. To give an even more precise characterization of Cat, we then prove Lurie's straightening–unstraightening theorem. Finally, we show how results from ∞-category theory can now be proved (e.g., various adjunctions between $\mathcal{S}$ and Cat) in short order and use the particular nature of type theory (in the form of the SHP) to construct new ∞-categories quickly and intuitively.

8.1 Related work

There has been a large amount of work on both directed approaches to type theory and straightening–unstraightening generally. Our work fits into the broader tradition of directed type theory, specifically the line of work initiated by [32] on simplicial type theory [3, 2, 4, 31, 37, 45, 44, 43, 10]. In this specific context, Cat fills a major missing piece in the foundations of STT and allows for new applications to, for instance, higher algebra and monoidal category theory. For directed type theory generally, we believe this construction should make it possible to more directly provide semantics to "fully directed" type theories [15, 40, 26, 25, 13, 27, 1, 24, 23, 22]. In particular, a model of those type theories which adequately models ∞-categories can now be constructed by a syntactic translation to $\mathsf{TT}_{\boxslash}$. The presence of Cat, in particular, means that the features supported by $\mathsf{TT}_{\boxslash}$ should be adequate to encode all of those presently considered.

This work also fits into the tradition of using the right adjoint to $\mathbb{I} \to -$ to construct a universal family with suitable structure. This dates back to [16], where it was used in the context of cubical type theory. While, for instance, [34] proposed a more refined version of this technique which gave a more judgmental account of the amazing right adjoint, our approach closely follows [16] and, especially, its adaptation to directed type theory by [41]. In particular, op. cit. use this methodology to construct a directed univalent universe of groupoids in bicubical type theory. [9] then introduced $\mathsf{TT}_{\boxslash}$ in order to adapt this methodology to apply to standard ∞-categories and further prove that the universe of groupoids is a category. Our work carries this program further forward by constructing not just a category of groupoids via a universal covariant family, but a category of categories via a universal cocartesian family. This is a substantial step – cocartesian families enjoy fewer nice properties than covariant families and this complicates various aspects of the proof e.g., the argument that Cat is simplicial.

Finally, since Lurie's original proof [17], two additional constructions of Cat have been published in the literature. The first is given by [11] and is a much more directed and succinct version of Lurie's proof, but still employs the same fundamental approach. In particular, the main result is a Quillen equivalence of two model categories, not an intrinsically ∞-categorical or synthetic approach. More closely related is the work of [6] which develops a universal cocartesian family as we do and uses this to prove straightening–unstraightening, directed univalence, and similar. The proof strategy of our approach is quite similar to op. cit.: a straightforward proof of the universal fibration and a more lengthy argument that its base is a category. In fact, op. cit. is influenced by various standard techniques in the semantics of type theory. A crucial difference between our approaches, however, is the ambient framework used to manage ∞-categories. Unlike our approach, [6] use quasicategories along with various model categorical tools from the Joyal and marked model structures. This is in contrast with our more synthetic approach, which uses model categories only indirectly via Theorem 11.

Particularly in light of the last proof, we feel that our approach is an interesting example in how higher category theory can influence the development of type theory (through HoTT) which can in turn contribute new techniques and proofs to higher category theory. We also hope that the model-independent nature of our proof will result in a result which can be applied to exotic situations such as internal ∞-categories in an ∞-topos [19, 20].

8.2 Future work

While this work has constructed the category of categories, much work still remains to be done around (1) the use of Cat in synthetic ∞-category theory and (2) deriving a computational account of this type theory, following on the work of [41]. For the first point, we believe that higher algebra and the theory of operads is particularly interesting to study in this regard, as the existing foundations of the theory [18] are notoriously technical and rely on intricate simplex-by-simplex arguments. For the second, we believe that the machinery of our argument could be adapted to bicubical type theory which would, at the very least, provide a constructive model of our argument.

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