From Co-Coverages to Radicals in Complete Lattices

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Abstract

Completeness and representation theorems in abstract algebra, lattice theory, and theoretical computer science are tied to the existence of ideal objects, and thus to transfinite methods such as Zorn's lemma. Yet many concrete uses of those theorems appeal to finite approximations only, which carry a clear computational meaning. In this paper, we introduce co-coverages on complete lattices as a uniform way to specify ideal elements, and associate to each co-coverage a canonical closure operator with a folding property reminiscent of the covering principles at work in constructive algebra. This yields finitary, choice-free alternatives for arguments in which ideal objects are employed to reduce a computational problem to subcases. In a classical setting, our closure operators admit bases which allow to recover a host of primality principles such as the universal Krull–Lindenbaum theorem and Henkin's lemma.

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1 Introduction

Motivation

A recurring proof pattern in lattice theory and abstract algebra is the use of prime, maximal, or otherwise "ideal" objects, e.g., to reduce a global argument to local cases. Typical examples include prime ideals in algebra and maximal consistent sets in completeness proofs. More often than not, such objects are brought into being by Zorn's lemma, employed within elegant existence results, yet offering little in the way of computational insight.

From a constructive point of view, such transfinite methods pose a difficult challenge: the objects whose existence is asserted are rarely computable, and the corresponding arguments cannot be directly interpreted as algorithms. Nevertheless, many of the uses of prime objects are finitary in nature, relying only on the ability to reason by cases or to "split" nondeterministic alternatives. In fact, from the point of view of contemporary constructive algebra [63], there is hardly any reason to climb into transfinite heights as long as our ends are computationally grounded [17, 18, 24, 63, 67, 83, 84, 86, 87, 99, 100].[1]

[1] This list of references is by no means meant to be exhaustive.

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Example

Subdirect products are part and parcel of universal algebra, and prominently so in the representation theories of groups, rings and modules. For instance, a cornerstone in the basic theory of commutative lattice-ordered groups (ℓ-groups) is Birkhoff's representation theorem [16, Theorem 3.6], according to which every ℓ-group is isomorphic to a subdirect product of *linearly ordered* ℓ-groups.[2] The practical import of Birkhoff's theorem is that we may reason "as if" an ℓ-group G were linearly ordered, say to prove an equation in G. To see how this works, let us take a step back from subdirect representations, and consider Birkhoff's prime ℓ-ideal principle [16, §20, Lemma 1], which anyway underpins the representation theorem: the only element common to every prime ideal is 0. Suppose now that a certain equation $\rho = 0$ indeed holds in every linearly ordered ℓ-group. To prove that this equation holds in G, it suffices to check that $\rho = 0$ modulo every prime ℓ-ideal P of A. But the quotients G/P are linearly ordered on account of primality!

Elegant as this may be, we are hard-pressed to re-establish the equation at hand by finitary means only. Lombardi and Quitté's *covering principle by quotients (for ℓ-groups)* [63, XI.2.10] answers for the lack of computational content of Birkhoff's theorem. Let us quote this result verbatim; let G be an ℓ-group, and let G^+ denote its set of non-negative elements:

▶ **Proposition** (Covering principle by quotients [63]). *Let $a, b \in G$, $x_1, \dots, x_n \in G^+$ with $\bigwedge_i x_i = 0$. Then $a \equiv b \mod x_i$ for each i if and only if $a = b$.*

Here, $x \equiv y \mod a$ iff there is an integer $n \geqslant 1$ for which $|x - y| \leqslant n|a|$ [63, XI.2.6]. For instance, since

$$(x - (x \wedge y)) \wedge (y - (x \wedge y)) = 0,$$

according to the Proposition it suffices to check a certain equation $a = b$ modulo either branch of

$$\begin{tikzcd}[column sep=small] & 0 = 0 \arrow[dl, dash] \arrow[dr, dash] & \\ x \wedge y = x & & x \wedge y = y \end{tikzcd}$$

which amounts to distinguishing the cases $x \leqslant y$ and $y \leqslant x$. In this manner, we can indeed do "as if" an ℓ-group were linearly ordered [63, p. 631]. A "slightly more abstract" way to put the above proposition, quoting again from [63, p. 622], is by saying that the canonical ℓ-group morphism

$$G \to \prod_i G/\mathcal{C}(x_i) \tag{1}$$

is injective, where $\mathcal{C}(x_i)$ consists of the group elements x congruent to 0 modulo x_i. Therefore, if $a \in \bigcap \mathcal{C}(x_i)$, then $a = 0$ in G. This is our guiding idea.

[2] The following discussion is closely related to that carried out in the Introduction of [66], where methods akin to those developed in the present paper are used to provide a constructive treatment of Mundici's Γ-functor. As a consequence, one obtains an equivalence between the category of *unital* ℓ-groups and the category of MV-algebras, a result that classically relies on Chang's subdirect representation theorem [22, Lemma 3].

Present approach

This paper develops a uniform framework in which splitting principles are treated directly, without appealing to Zorn's lemma. Our approach is based on the notion of a co-coverage $\mathcal{C}$ on a complete lattice L, which specifies how an element may be decomposed. Co-coverages dualize the coverings from point-free topology [57, II.2.11] and capture, in a single formalism, branching rules such as those characteristic for prime ideals and nondeterministic computations. To each co-coverage $\mathcal{C}$ we associate a closure operator $j_{\mathcal{C}}$. The defining property of $j_{\mathcal{C}}$ is that it "folds up branchings", i.e., for every $x \in L$ and $S \in \mathcal{C}(x)$,

$$j_{\mathcal{C}}(x) = \bigwedge_{s \in S} j_{\mathcal{C}}(s). \tag{2}$$

This property reflects the constructive, computational core of many, if not all, classical prime intersection principles.

Main contributions

1. We introduce co-coverages on complete lattices as a uniform way to capture nondeterminism arising in logic, point-free topology, and algebra.
2. We associate to each co-coverage $\mathcal{C}$ a canonical closure operator $j_{\mathcal{C}}$ that transforms co-covers into meets and is optimal with this property.
3. With the axiom of choice (AC) we show that $j_{\mathcal{C}}$ has a base of ideal elements that recovers completeness results.
4. We show that AC and classical logic are necessary for the spectral representation of $j_{\mathcal{C}}$.

Related work

A general theory of radicals in complete lattices was proposed early on by Amitsur [4–6] and Isphording [56], though geared towards providing an axiomatic foundation for the various radicals at work in the structure theory of rings and modules. By contrast, the recent radical-theoretic treatment of the Teichmüller–Tukey lemma [67], along with the related proposals for a Jacobson radical in logic [38, 43, 85], echo in the present paper's attempt to give constructive meaning to several consequences of the axiom of choice beyond those formerly treated [84]. The present work may indeed be understood as an extension of [67, 84]. Whereas [84] focuses on prime ideal theorems related to *finitary* nondeterministic axioms, the framework developed here removes this restriction by further pursuing the approach initiated in [67]. Conceptually, the radical theory of Scott-open filters offered in [67] is now subsumed into a theory of closure operators canonically associated with co-coverages. The price paid is that we again adopt an impredicative setting (**IZF**), thereby sacrificing a certain degree of generality, in marked contrast with the approach of [84]. Nevertheless, sufficiently concrete situations still offer some leeway [67, Prop. 15], though it remains a challenge to recast the theory in a predicative setting such as **CZF** [2, 3, 74].

Related in spirit is recent work on constructive maximal and prime ideals, which reinterprets classical prime constructions via forcing methods [19, 20], thus clarifying how prime and maximal ideals can be viewed as heuristic devices rather than actual objects, a perspective complementary to the present radical-theoretic approach. Finally, our work follows the tradition of contemporary constructive algebra [25, 26, 29, 30, 62, 63, 100][3] in pursuing a point-free, lattice-theoretic approach to the transfinite methods pervasive in abstract algebra.

[3] Again, this list of references should be understood as selective.

Structure of this paper

Order-theoretic preliminaries will be pinned down in the following section. In Section 3 we introduce the key concepts, among which the radical associated to a Scott-open set from [67], and co-coverages, dualizing Johnstone's coverages [57, II.2.11], along with the related ideals, which generalize Erné's distributors [36, 37]. In Section 3.5 we associate to every co-coverage $\mathcal{C}$ a corresponding closure operator $j_{\mathcal{C}}$. In Section 4 we turn our attention to the intended semantics of co-coverages, as specifying "ideal elements" in a rather general sense. We show that the spectrum of a co-coverage forms a basis of fixed elements for $j_{\mathcal{C}}$ (Proposition 31). Most subsections conclude with examples that either illustrate a definition or instantiate a result. Some more substantial applications will be offered in Section 5. We discuss extensions of the finitary coverings at work in formal topology [80]. In particular, we provide a radical-theoretic conservativity criterion (Proposition 41) which applies, e.g., to the Universal Krull–Lindenbaum Theorem [75]. Moreover, we discuss a new and related approach to Henkin's lemma (Proposition 43), and we present an application to nondeterministic computations in Section 5.4. Last but not least, while the spectral representation (Proposition 31) requires a classical framework, in Section 6 we address some of the extent to which AC and the law of excluded middle (LEM) are necessary.

2 Preliminaries

On method and foundations

The present paper is primarily set within the constructive but impredicative Intuitionistic Zermelo–Fraenkel Set Theory **IZF** [10, 11, 13, 15, 32, 45, 82]. In particular, as opposed to predicative [31] settings such as Constructive Zermelo–Fraenkel Set Theory **CZF** [2, 3, 32, 74], we have at our disposal both the power set axiom and the scheme of full separation. By a *finite* set we understand a set that can be written in the form $\{ a_1, \ldots, a_n \}$ for some $n \geqslant 0$.[4] Every finite set is either empty or inhabited. We denote by $\mathrm{Fin}(S)$ the set of all finite subsets of a given set S. To compare (and derive) some new forms of AC, we sometimes move to **ZF(C)**. Any such deviation from the otherwise constructive framework will be highlighted.

Complete lattices

We recall several standard order-theoretic notions from [48, 51, 57]. A *complete lattice* is a partially ordered set L in which every subset N has both a join $\bigvee N$ and a meet $\bigwedge N$. In particular, a complete lattice has top and bottom elements $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$, respectively. A subset D of L is *directed* if D is inhabited and such that, for every $x, y \in D$, there is $z \in D$ with $x \leqslant z$ and $y \leqslant z$. A subset O of L is *open* if, for all directed $D \subseteq L$,

$$\bigvee D \in O \quad \text{implies} \quad D \between O,$$

where from formal topology [80] we borrow the *overlap* symbol: $D \between O$ means that $D \cap O$ is inhabited (see also [70, p. 579]). The *Scott-open* subsets of L are the *upward closed* open subsets, i.e., those open subsets O for which

$$x \in O \text{ and } x \leqslant y \quad \text{implies} \quad y \in O.$$

[4] The list $a_1, \ldots, a_n$ may contain duplicates. Thus, following [76], we again deviate from the prevalent terminology of constructive mathematics [3, 63, 65] that calls *finitely enumerable* a finite set in the sense above, i.e., a set S for which there is a surjection from $\{ 1, \ldots, n \}$ to S for some $n \geqslant 0$; and that reserves the term *finite* to sets which are in bijection with $\{ 1, \ldots, n \}$ for a necessarily unique $n \geqslant 0$.

For instance, with LEM, the set-theoretic complements of principal downsets, $({\downarrow}y)^c = L - {\downarrow}y$, are Scott-open in complete lattices.

A function $f : L \to L$ is said to be *Scott-continuous* if it preserves *directed* joins:

$$f\left(\bigvee D\right) = \bigvee \{\, f(d) \mid d \in D \,\}\,.$$

An element a of L is *compact* or *finite* [57, 96] if, for all directed subsets D of L such that $a \leqslant \bigvee D$, there is $d \in D$ with $a \leqslant d$. Thus, a is compact precisely when the principal filter

$${\uparrow}a = \{\, b \in L \mid a \leqslant b \,\}$$

is Scott-open. The subset of compact elements of L is often denoted by KL. We say that L is *compact* if $1 \in KL$.

A complete lattice L is said to be *algebraic* if every element x of L is the join of compact elements $a \leqslant x$, that is,

$$x = \bigvee \{\, a \in KL \mid a \leqslant x \,\} = \bigvee (KL \cap {\downarrow}x).$$

The prime example of an algebraic lattice is the power set $L = \mathrm{Pow}(S)$ of a set S, for which $KL = \mathrm{Fin}(S)$.

Closure operators

Let L be a complete lattice. A *closure operator* on L is a function $f : L \to L$ which is *extensive*, *monotone*, and *idempotent*. This is to say that $x \leqslant f(x)$, and $f(x) \leqslant f(y)$ whenever $x \leqslant y$, and $f(f(x)) \leqslant f(x)$, respectively, for every $x, y \in L$. Recall that the closure properties boil down to demanding, for every $x, y \in L$, that $x \leqslant f(y)$ iff $f(x) \leqslant f(y)$. If f is a closure operator on L, then we write

$$L_f = \{\, x \in L \mid f(x) = x \,\}$$

for the corresponding set of fixed points. This L_f is again a complete lattice, where

$$\bigwedge_{L_f} N = \bigwedge N \quad \text{and} \quad \bigvee_{L_f} N = f\left(\bigvee N\right).$$

A closure operator f on L is *algebraic* if

$$f(x) = \bigvee \{\, f(a) \mid a \in KL \cap {\downarrow}x \,\}\,.$$

In an algebraic lattice, the algebraic closure operators are precisely the Scott-continuous ones. A *basis* for a closure operator f on L is a set $P \subseteq L$ such that $f(x) = \bigwedge P/x$ for every $x \in L$, where we borrow the slice category notation to write $P/x = P \cap {\uparrow}x$. Trivially, every closure operator has at least the basis L_f of its fixed points. We write

$$\mathrm{Clos}(L)$$

for the set of all closure operators on L, which we order pointwise, viz.,

$$f \leqslant g \quad \text{iff} \quad (\forall x \in L)\, f(x) \leqslant g(x).$$

Top-kernels of $f \in \mathrm{Clos}(L)$ will be written

$$\nabla(f) = f^{-1}(1) = \{\, x \in L \mid f(x) = 1 \,\}\,.$$

The following lemma will be made use of repeatedly.

▶ **Lemma 1.** *Let $f \in \mathrm{Clos}(L)$. For every $x, y \in L$,*

$$f(x \vee y) = f(f(x) \vee y).$$

Proof. On the one hand, $x \vee y \leqslant f(x) \vee y$ since f is extensive, so that $f(x \vee y) \leqslant f(f(x) \vee y)$ by monotonicity. On the other hand, $f(x) \leqslant f(x \vee y)$ and $y \leqslant x \vee y \leqslant f(x \vee y)$, hence $f(x) \vee y \leqslant f(x \vee y)$, which yields $f(f(x) \vee y) \leqslant f(f(x \vee y)) = f(x \vee y)$ since f is idempotent. ◀

3 Radical operators

3.1 Radicals of Scott-open sets

Throughout Section 3 and Section 4, let L be a fixed but otherwise arbitrary complete lattice, if not explicitly instantiated within an example. Let $O \subseteq L$, and $x \in L$. We consider

$$O : x = \{ y \in L \mid x \vee y \in O \},$$

and write

$$x \sqsubseteq_O y \quad \text{as shorthand for} \quad (O : x) \subseteq (O : y).$$

Suppose now that $O \subseteq L$ is Scott-open. From [67] we borrow the *radical* of O, which is nothing but the corresponding *saturation map* [37]

$$j_O : L \to L, \quad j_O(y) = \bigvee J_O(y), \tag{3}$$

where $J_O(y) = \{ x \in L \mid x \sqsubseteq_O y \}$. This j_O in fact is a closure operator [67, Proposition 10]. Notice also that $J_O(y)$ is directed [67, Lemma 8.3], which implies that j_O is *co-dense* with respect to O [67, Proposition 11], i.e., for every $y \in L$,

$$y \in O \quad \text{iff} \quad j_O(y) \in O. \tag{4}$$

The following result, proved and discussed in [67, Proposition 20], serves as a point of departure for the present development.

▶ **Proposition 2.** *Let $O \subseteq L$ be Scott-open. Among closure operators on L, the radical j_O is uniquely determined by the following properties:*

(i) $O = \nabla(j_O)$*,*

(ii) j_O *folds* up branchings*, i.e., for every $x, a \in L$,*

$$j_O(x) = j_O(x \vee a) \wedge \bigwedge_{b \in (O:a)} j_O(x \vee b). \tag{5}$$

More precisely, if f is a closure operator, then $O = \nabla(f)$ implies $f \leqslant j_O$, or equivalently $f j_O = j_O$; if in addition f folds up branchings, then $f = j_O$.

We will also rely on the following auxiliary result [67, Lemma 14].

▶ **Lemma 3.** *For every $x, y \in L$, the following are equivalent.*

(i) $x \in J_O(y)$*,*

(ii) $x \leqslant j_O(y)$*,*

(iii) $x \sqsubseteq_O y$*.*

▶ **Example 4.** Recall from [93] that an element $x \in L$ is said to be *small* if $x \vee y \neq 1$ for every $y \neq 1$. Put contrapositively, this is the case precisely when $x \sqsubseteq_{\{1\}} 0$, so that $j_{\{1\}}(0)$ amounts by (3) to the join of all small elements (in the latter sense). With AC, the radical element $j_{\{1\}}(0)$ equals the meet of dual atoms of L, for which see Example 33 later on.

3.2 Co-coverages

We work with co-coverages, i.e., coverages [57, II.2.11] on the dual lattice L^{op}, to represent the admissible branchings above any given lattice element. Let us make this explicit.

▶ **Definition 5.** *By a* co-coverage *on L we understand a function $\mathcal{C} : L \to \mathrm{Pow}(\mathrm{Pow}(L))$ which to every $x \in L$ assigns a* dominating *and* join-stable *family of subsets of L, which is to say that*
(i) *$S \subseteq {\uparrow}x$ for every $S \in \mathcal{C}(x)$, and*
(ii) *$S \in \mathcal{C}(x)$ & $x \leqslant y \implies S \vee y \in \mathcal{C}(y)$,*
where $S \vee y = \{ s \vee y \mid s \in S \}$. The collection of co-coverages on L is partially ordered by stipulating, pointwise,

$$\mathcal{C} \subseteq \mathcal{C}' \quad \textit{iff} \quad (\forall x \in L)\, \mathcal{C}(x) \subseteq \mathcal{C}'(x). \tag{6}$$

We can generate co-coverages from sets of "initial" co-covers, to which end we employ a somewhat different terminology, which derives from van den Berg's *nondeterministic inductive definitions* [94]. (Compare also Definition 24 below with [94, Definition 2.2], but mind that we are working in an impredicative setting.)

▶ **Definition 6.** *A* nondeterministic axiom *over L is a pair $(a, B) \in L \times \mathrm{Pow}(L)$.*

▶ **Lemma 7.** *Let $\mathcal{E}$ be a set of nondeterministic axioms over L. There is a co-coverage $\mathcal{C}_{\mathcal{E}}$ such that $B \vee a \in \mathcal{C}_{\mathcal{E}}(a)$ whenever $(a, B) \in \mathcal{E}$. Among co-coverages with this property, $\mathcal{C}_{\mathcal{E}}$ is the least one with respect to the pointwise ordering* (6) *on co-coverages on L.*

Proof. For every $x \in L$, let

$$\mathcal{C}_{\mathcal{E}}(x) = \{ B \vee x \mid (a, B) \in \mathcal{E} \text{ and } a \leqslant x \}. \tag{7}$$

It is easy to see that $\mathcal{C}_{\mathcal{E}}$ satisfies the conditions of Definition 5. Clearly $B \vee a \in \mathcal{C}_{\mathcal{E}}(a)$ for every axiom $(a, B) \in \mathcal{E}$. Now consider any other co-coverage $\mathcal{C}$ on L with the latter property. To show that $\mathcal{C}_{\mathcal{E}} \subseteq \mathcal{C}$, let $x \in L$ and $S \in \mathcal{C}_{\mathcal{E}}(x)$. Then $S = B \vee x$ for some $(a, B) \in \mathcal{E}$ with $a \leqslant x$. By the assumption, $B \vee a \in \mathcal{C}(a)$. Because $\mathcal{C}$ is join-stable, it follows that $(B \vee a) \vee x = B \vee x = S \in \mathcal{C}(x)$. ◀

Notice that if $(a, B) \in \mathcal{E}$ is such that $B \subseteq {\uparrow}a$, then $B = B \vee a \in \mathcal{C}_{\mathcal{E}}(a)$.

3.3 Folding up

To reason uniformly across branches, we need closure operators that respect the branching structure imposed by a co-coverage.

▶ **Definition 8.** *Let $\mathcal{C}$ be a co-coverage on L, and let $f \in \mathrm{Clos}(L)$. We say that f* folds up *$\mathcal{C}$, if it transforms co-covers to meets, i.e., for every $x \in L$ and $S \in \mathcal{C}(x)$,*

$$f(x) = \bigwedge_{s \in S} f(s).$$

Notice that $f(x) \leqslant \bigwedge_{s \in S} f(s)$ is always the case, because $f(x) \leqslant f(x \vee s) = f(s)$ for every $s \in S \in \mathcal{C}(x)$.

In the following, if f is a closure operator on L, then we write

$$f/x := L_f/x = L_f \cap {\uparrow}x = L_f \cap {\uparrow}f(x),$$

where the (third) equality holds by way of the closure properties of f.

▶ **Proposition 9.** *Let $\mathcal{C}$ be a co-coverage on L, and let $f \in \mathrm{Clos}(L)$. The following are equivalent.*

(i) *f folds up $\mathcal{C}$.*

(ii) *For every $x \in L$ and $S \in \mathcal{C}(x)$, the canonical mapping to the direct product,*

$$e : f/x \to \prod_{s \in S} f/(x \vee s), \quad e(y) : s \mapsto f(y \vee s), \tag{8}$$

is an order embedding (order-preserving and reflecting).

Proof. (i) $\Rightarrow$ (ii) The mapping in question is certainly monotone. To show that it reflects, let $y, y' \in f/x$ and suppose that $e(y) \leqslant e(y')$. This amounts to $f(y \vee s) \leqslant f(y' \vee s)$ for every $s \in S$. Since $\mathcal{C}$ is join-stable, we know that $S \vee y \in \mathcal{C}(y)$ and $S \vee y' \in \mathcal{C}(y')$, and because f folds up $\mathcal{C}$, we obtain $y = f(y) = \bigwedge_{s \in S} f(y \vee s) \leqslant \bigwedge_{s \in S} f(y' \vee s) = f(y') = y'$.

(ii) $\Rightarrow$ (i) Suppose that (8) is an embedding for every $x \in L$ and $S \in \mathcal{C}(x)$. To see that f folds up $\mathcal{C}$, let $x \in L$ and $S \in \mathcal{C}(x)$. It suffices to check that $\bigwedge_{s \in S} f(s) \leqslant f(x)$. To this end, note first that $y := \bigwedge_{s \in S} f(s) \in f/x$. For every $s_0 \in S$ we compute $e(y)(s_o) = f(y \vee s_o) = f((\bigwedge_{s \in S} f(s)) \vee s_0) \leqslant f(f(s_0) \vee s_0) = f(f(s_0)) = f(s_0) = f(x \vee s_0) = f(f(x) \vee s_0) = e(f(x))(s_0)$ according to Lemma 1 and the closure properties of f. Therefore $e(y) \leqslant e(f(x))$, and thus $y \leqslant f(x)$, as required. ◀

Discussion

Suppose that $f \in \mathrm{Clos}(L)$ folds up some $\mathcal{C}$ on L. It follows from Proposition 9 that $a \leqslant f(b)$ already if there is $S \in \mathcal{C}(a)$ such that $a \leqslant f(b \vee s)$ for every $s \in S$. In fact, under this assumption we obtain from the closure properties and Lemma 1 that, for every $s \in S$,

$$f(f(a) \vee s) = f(a \vee s) \leqslant f(b \vee s) = f(f(b) \vee s),$$

which is to say that $e(f(a)) \leqslant e(f(b))$ for the corresponding embedding e of Proposition 9, and therefore $f(a) \leqslant f(b)$, which in turn is equivalent to $a \leqslant f(b)$. This is particularly useful in the case in which f is *grounded* [91], i.e., whenever $f(0) = 0$, for then the question whether $a = 0$ can be settled modulo suitable branchings. The *folding* property (2) thus mimics the requirement (1) in the setting of complete lattices.

The question at stake is whether the folding property can be satisfied nontrivially. For instance, the "collapsing" closure $f \equiv 1$ folds up any co-coverage $\mathcal{C}$ whatsoever. Whether there is to any given co-coverage a corresponding non-collapsing closure operator, transforming co-covers to meets, depends on the existence of non-trivial open $\mathcal{C}$-distributors (Proposition 16(ii)), to which we turn our attention in the following section.

▶ **Example 10.** Let R be a commutative ring with 1, and let L be the complete (algebraic) lattice of ideals $\mathfrak{a}$ of R, ordered by set-inclusion. We write $\langle T \rangle$ for the ideal generated by a subset T of R, and abbreviate the principal ideal $\langle \{ a \} \rangle$ with $\langle a \rangle$. In L, the meet of ideals is given by set-intersection, and the join of a family of ideals $\mathfrak{a}_i$ $(i \in I)$ is given by $\langle \bigcup_{i \in I} \mathfrak{a}_i \rangle$.[5] We consider the co-coverage $\mathcal{C}$ which is generated according to Lemma 7 by the set of all (nondeterministic) *primality axioms*, viz.,

$$\big(\langle ab \rangle, \{ \langle a \rangle, \langle b \rangle \} \big) \quad \text{for every } a, b \in R.$$

[5] See Section 5.1 below for a related and more general approach in terms of covering relations.

What does it mean for a certain closure operator f on L to fold up this $\mathcal{C}$? First, by the construction of $\mathcal{C}$, for every $a, b \in R$ we have that $S = \{ \langle a \rangle, \langle b \rangle \} \vee \langle ab \rangle \in \mathcal{C}(\langle ab \rangle)$. Since $\langle xy \rangle \subseteq \langle x \rangle$ for every $x, y \in R$, join-stability (Definition 5(ii)) gives that, for every $\mathfrak{a} \in L$,

$$S \vee \mathfrak{a} = \{ \langle a \rangle \vee \langle ab \rangle \vee \mathfrak{a}, \langle b \rangle \vee \langle ab \rangle \vee \mathfrak{a} \} = \{ \langle a \rangle \vee \mathfrak{a}, \langle b \rangle \vee \mathfrak{a} \} \in \mathcal{C}(\langle ab \rangle \vee \mathfrak{a}),$$

which would have to be transformed into a meet by the f in question. This requirement is indeed met by the *nilradical* $\sqrt{-}$, which to every ideal $\mathfrak{a}$ associates the set of ring elements some power of which is in $\mathfrak{a}$,

$$\sqrt{\mathfrak{a}} = \{ a \in R \mid (\exists n > 0)\, a^n \in \mathfrak{a} \} .$$

The nilradical is a closure operator on L, with the well-known (though seldom made explicit) property that it folds up the primality axiom:

$$\sqrt{\mathfrak{a} \vee \langle ab \rangle} = \sqrt{\mathfrak{a} \vee \langle a \rangle} \cap \sqrt{\mathfrak{a} \vee \langle b \rangle}. \tag{9}$$

Oftentimes, (9) suffices for a constructive proof of classical results that hinge on *Krull's lemma* (KL) [83, 86, 97]. Recall that KL asserts that $\sqrt{\langle 0 \rangle}$ is the intersection of all prime ideals [7, Proposition 1.8]. KL is classically equivalent to the Boolean prime ideal theorem [54].

3.4 Distributors

To construct the canonical folding closure operator $j_{\mathcal{C}}$, we identify those properties that are stable under all branchings. The key is Erné's concept of a *distributor* [36, 37], suitably generalized as follows.

▶ **Definition 11.** *Let $\mathcal{C}$ be a co-coverage on L. By a $\mathcal{C}$-*distributor *we understand an upward closed $U \subseteq L$, which in addition is* closed under co-covers*, i.e., such that, for every $x \in L$ and $S \in \mathcal{C}(x)$,*

$$S \subseteq U \quad \Longrightarrow \quad x \in U.$$

By upward closure this implication in fact is an equivalence.

In other words, the distributors in the sense of Definition 11 are nothing but the duals of Johnstone's $\mathcal{C}$*-ideals* for *coverages* on meet-semilattices [57, II.2.11].

▶ **Example 12.** Recall that a filter $F \subseteq L$ is said to be (meet-)*complete* if $\bigwedge B \in F$ whenever $B \subseteq F$, with no restriction whatsoever on B. The complete filters of L are precisely the $\mathcal{C}_{\mathcal{E}}$-distributors for $\mathcal{E} = \{ (\bigwedge B, B) \mid B \in \mathrm{Pow}(L) \}$.

▶ **Example 13.** Suppose that L is multiplicative, i.e., equipped with an order-preserving binary operation $\cdot : L \times L \to L$. In this setting, Erné's distributors [37] are the inhabited upward closed subsets U of L such that, for every $a, b, c \in L$,

$$a \vee c \in U \text{ and } b \vee c \in U \quad \Longleftrightarrow \quad a \cdot b \vee c \in U. \tag{10}$$

For the sake of simplicity, suppose that $\cdot$ is commutative, as well as unital in that $a \cdot 1 = a$. Since $b \leqslant 1$ anway, and because the multiplication is supposed to be order-preserving, it follows under these assumptions that $a \cdot b \leqslant a$ for every $a, b \in L$. If we now stipulate that $\mathcal{E}$ consist of all the instances of

$$(a_1 \cdots a_k, \{ a_1, \dots, a_k \}) \quad (k \geqslant 0), \tag{11}$$

then the corresponding $\mathcal{C}_{\mathcal{E}}$-distributors are precisely Erné's distributors (10).

▶ **Example 14.** Let L be the dual of the division lattice of natural numbers, ordered by stipulating $n \preccurlyeq m$ iff $m \mid n$ (i.e., m divides n), and where

$$n \vee m = \gcd(m,n) \quad \text{and} \quad n \wedge m = \operatorname{lcm}(m,n)$$

are given by the greatest common divisor, and least common multiple, respectively. This L is multiplicative with the usual multiplication on natural numbers. Consider the corresponding axiom set $\mathcal{E}$ (11) of Example 13. What are the $\mathcal{C}_{\mathcal{E}}$-distributors? To answer this, let $\mathbb{P}$ denote the set of prime numbers. According to Erné [37, p. 6 (3)], for every $M \subseteq \mathbb{P}$, the set

$$D(M) = \{\, n \in \mathbb{N} \mid (\forall p \in \mathbb{P})(n \preccurlyeq p \to p \in M) \,\}$$

is a $\mathcal{C}_{\mathcal{E}}$-distributor. We claim that these are precisely the $\mathcal{C}_{\mathcal{E}}$ distributors. To see this, consider an arbitrary $\mathcal{C}_{\mathcal{E}}$-distributor U. Then $U = D(M)$ for $M = U \cap \mathbb{P}$. In fact, on the one hand, if $n \in U$, consider any prime factor $p \mid n$. Then $p \in U$ by upward closure of U, hence $p \in M$, as required for $n \in D(M)$. Regarding the converse, consider $n \in D(M)$, and write n as a product of prime numbers $n = p_1 \cdots p_k$. Since $n \preccurlyeq p_i$, we know that $p_1, \ldots, p_k \in M \subseteq U$. Because U is a $\mathcal{C}_{\mathcal{E}}$-distributor, and since $(n, \{\, p_1, \ldots, p_k \,\}) \in \mathcal{E}$, it follows that $n \in U$.

3.5 Radicals from co-coverages

In this section, we associate to every co-coverage on L a corresponding closure operator $j_{\mathcal{C}}$ which folds up $\mathcal{C}$ and collapses precisely when L is the only Scott-open distributor for the $\mathcal{C}$ at play.

Here is the key concept of this paper.

▶ **Definition 15.** *Let $\mathcal{C}$ be a co-coverage on L, and let*

$$\mathrm{SOD}(\mathcal{C})$$

denote the collection of all Scott-open *$\mathcal{C}$-distributors. Let us write*

$$x \sqsubseteq_{\mathcal{C}} y \quad \textit{to abbreviate} \quad (\forall O \in \mathrm{SOD}(\mathcal{C}))\, x \sqsubseteq_O y,$$

where $\sqsubseteq_O$ is defined as in Section 3.1. We consider the mapping

$$j_{\mathcal{C}} : L \to L, \quad j_{\mathcal{C}}(y) = \bigvee \{\, x \in L \mid x \sqsubseteq_{\mathcal{C}} y \,\}, \tag{12}$$

and say that this $j_{\mathcal{C}}$ is the $\mathcal{C}$-radical.

▶ **Proposition 16.** *Let $\mathcal{C}$ be a co-coverage on L. The $\mathcal{C}$-radical is a closure operator, enjoying the following properties:*

(i) $j_{\mathcal{C}}(x) = \bigwedge_{O \in \mathrm{SOD}(\mathcal{C})} j_O(x)$,

(ii) $\nabla(j_{\mathcal{C}}) = \bigcap \mathrm{SOD}(\mathcal{C})$,

(iii) *$j_{\mathcal{C}}$ is* co-dense *with respect to $\mathrm{SOD}(\mathcal{C})$ in the sense that, for every $x \in L$ and every $O \in \mathrm{SOD}(\mathcal{C})$,*

$$x \in O \quad \textit{iff} \quad j_{\mathcal{C}}(x) \in O,$$

(iv) *j_C folds up $\mathcal{C}$.*

Moreover, if f is a closure operator on L, then $f \leqslant j_{\mathcal{C}}$ already if f is co-dense with respect to $\mathrm{SOD}(\mathcal{C})$.

Proof. Every meet of closure operators is again a closure operator, which for $j_{\mathcal{C}}$ is the case by (i). We go through the items one after another.

(i) For every $O \in \mathrm{SOD}(\mathcal{C})$, we know that $x \sqsubseteq_O y$ iff $x \leqslant j_O(y)$ according to Lemma 3. Hence, with (3),

$$j_{\mathcal{C}}(y) = \bigvee \left\{ x \in L \;\middle|\; x \leqslant \textstyle\bigwedge_{O \in \mathrm{SOD}(\mathcal{C})} j_O(y) \right\} = \bigwedge_{O \in \mathrm{SOD}(\mathcal{C})} j_O(y).$$

(ii) With (i) we see that $j_{\mathcal{C}}(x) = 1$ precisely when $j_O(x) = 1$ for every $O \in \mathrm{SOD}(\mathcal{C})$, which by Proposition 2(i) is to say that $x \in \bigcap_{O \in \mathrm{SOD}(\mathcal{C})} \nabla(j_O) = \bigcap \mathrm{SOD}(\mathcal{C})$.

(iii) The left-to-right implication is due to the upward closure of Scott-opens. For the converse, let $O \in \mathrm{SOD}(\mathcal{C})$ and suppose that $j_{\mathcal{C}}(x) \in O$. By (i) and upward closure again, it follows that $j_O(x) \in O$. The co-density of j_O (4) yields $x \in O$.

(iv) Let $x \in L$ and $S \in \mathcal{C}(x)$. To show that $j_{\mathcal{C}}$ folds up $\mathcal{C}$, it suffices by (i) to check that so does every j_O for $O \in \mathrm{SOD}(\mathcal{C})$. By monotonicity, and since $x \leqslant s$ for every $s \in S$, it suffices to concentrate on showing that $\bigwedge_{s \in S} j_O(s) \leqslant j_O(x)$. According to Lemma 3, this is the case precisely when $\bigwedge_{s \in S} j_O(s) \sqsubseteq_O x$. Thus, let $z \in L$ such that $(\bigwedge_{s \in S} j_O(s)) \vee z \in O$. We need to show that $x \vee z \in O$. By the upward closure of O, we have that $j_O(s) \vee z \in O$ for every $s \in S$. Lemma 1 implies that $j_O(j_O(s) \vee z) = j_O(s \vee z) \in O$ for every $s \in S$. Because j_O is co-dense with respect to O, it follows that $s \vee z \in O$, but then also $s \vee z \vee x \in O$, still for every $s \in S$. Since $\mathcal{C}$ is a co-coverage, we know that $S' = \{ s \vee (x \vee z) \mid s \in S \} \in \mathcal{C}(x \vee z)$. Then, because O is a $\mathcal{C}$-distributor and $S' \subseteq O$, we obtain $x \vee z \in O$.

As regards the add-on, consider some $f \in \mathrm{Clos}(L)$ which is co-dense with respect to every Scott-open $\mathcal{C}$-distributor. To check that $f \leqslant j_{\mathcal{C}}$, consider some $O \in \mathrm{SOD}(\mathcal{C})$. Now, for every $x, z \in L$, if $f(x) \vee z \in O$, then, using upward closure and Lemma 1, it follows that $f(f(x) \vee z) = f(x \vee z) \in O$, hence $x \vee z \in O$ by co-density. This is to say that $f(x) \sqsubseteq_O x$. Lemma 3 yields $f \leqslant j_O$, for every $O \in \mathrm{SOD}(\mathcal{C})$, so that $f \leqslant j_{\mathcal{C}}$ by (i). ◀

▶ **Proposition 17.** *Suppose that L is algebraic, and let f be an algebraic closure operator on L. If f folds up $\mathcal{C}$, then $j_{\mathcal{C}} \leqslant f$.*

Proof. Suppose that f is an algebraic closure operator on L which folds up $\mathcal{C}$. Under this assumption, we claim that, for every $a \in KL$,

$$D_a = \{ y \in L \mid a \leqslant f(y) \} \in \mathrm{SOD}(\mathcal{C}).$$

To verify that D_a is open, let $X \subseteq L$ be directed and suppose that $\bigvee X \in D_a$. This is to say that $a \leqslant f(\bigvee X)$, and thus $a \leqslant \bigvee_{x \in X} f(x)$ since f is continuous. Because a is supposed to be compact, there is $x \in X$ such that $a \leqslant f(x)$, hence $X \between D_a$. Upward closure of D_a is clear. Next, to see that D_a is a $\mathcal{C}$-distributor, consider $x \in L$ and $S \in \mathcal{C}(x)$, and suppose that $S \subseteq D_a$. This is to say that $a \leqslant f(s)$ for every $s \in S$, and hence $a \leqslant \bigwedge_{s \in S} f(s) = f(x)$, because f folds up $\mathcal{C}$. Therefore $x \in D_a$, as required.

Next we show that under the above assumptions it follows that $j_{\mathcal{C}} \leqslant f$. To this end, it suffices by (12) to show that $x \leqslant f(y)$ whenever $x \sqsubseteq_{\mathcal{C}} y$. If the latter is indeed the case, then we know that in particular

$$x \sqsubseteq_{D_a} y \quad \text{for every } a \in KL. \tag{13}$$

Now consider any $a \in KL \cap {\downarrow} x$. Since $a \leqslant x \leqslant f(x) = f(x \vee 0)$, we obtain from (13) that $a \leqslant f(y \vee 0) = f(y)$. It follows that $x = \bigvee(KL \cap {\downarrow} x) \leqslant f(y)$, as was to be shown. ◀

▶ **Definition 18.** *If $f : L \to L$ is a function, let $\mathcal{C}_f$ denote the co-coverage on L that is generated according to Lemma 7 by $\{ (x, \{ f(x) \}) \mid x \in L \}$.*

▶ **Corollary 19.** *Suppose that L is algebraic. Every algebraic closure operator f on L is of the form $f = j_{\mathcal{C}_f}$.*

Proof. Let $\mathcal{C}_f$ be according to Definition 18. Since $j_{\mathcal{C}_f}$ transforms co-covers to meets, we know that $j_{\mathcal{C}_f}(x) = j_{\mathcal{C}_f}(f(x))$ for every $x \in L$, and because $j_{\mathcal{C}_f}$ is extensive, we see that $f(x) \leqslant j_{\mathcal{C}_f}(x)$. Thus $f \leqslant j_{\mathcal{C}_f}$. But clearly f transforms co-covers to meets as well, simply because $f(x) = f(f(x))$, so $j_{\mathcal{C}_f} \leqslant f$ by Proposition 17. ◀

With AC, we can swiftly extend Corollary 19 to every closure operator whatsoever on an arbitrary complete lattice L; see Corollary 32 below.

▶ Remark 20. It is not unprecedented to refer to closure operators on complete lattices as "radicals" [60, (2.13)]. This terminology is supported by Amitsur's general theory of radicals [4–6], which provides an axiomatic framework for the radical theory arising in the structural study of rings, modules, and algebras, and has since led to further general developments [46]. Our radicals fit into Amitsur's framework by way of [60, Theorem 3.6].

3.6 Examples

▶ **Example 21.** Let $\vdash_i$ and $\vdash_c$ stand, respectively, for derivability in intuitionistic and classical logic in a propositional language S. It is known that

$$\Gamma \vdash_c \varphi \quad \text{iff} \quad \Gamma, \psi_1 \vee \neg\psi_1, \ldots, \psi_n \vee \neg\psi_n \vdash_i \varphi \tag{14}$$

for certain propositional variables ψ_i occurring in Γ or φ, see, e.g., [55] and [68, p. 27]. Now we consider the (algebraic) lattice L of $\vdash_i$-theories, i.e., the deductively closed sets of formulas, ordered by set-inclusion. Consider the co-coverage $\mathcal{C}$ which is generated by all instances of the following axiom

$$(\langle \top \rangle, \{ \langle \psi \rangle, \langle \neg\psi \rangle \}), \tag{15}$$

where we write $\langle \Gamma \rangle$ for the deductive closure (with respect to $\vdash_i$) of a set Γ of formulas. The corresponding radical yields another proof of

▶ **Glivenko's theorem.** $\Gamma \vdash_c \varphi$ *iff* $\Gamma \vdash_i \neg\neg\varphi$.

Proof. Write $f(\Gamma) = \{ \chi \mid \Gamma \vdash_i \neg\neg\chi \}$, which yields an algebraic closure operator on L. This f transforms co-covers to meets, and Proposition 17 implies that $j_{\mathcal{C}}(\Gamma) \subseteq f(\Gamma)$. Now, if $\Gamma \vdash_c \varphi$, then $\varphi \in \bigwedge j_{\mathcal{C}}(\Gamma, \pm\psi_1, \ldots, \pm\psi_n)$ by (14), where the meet ranges over the possible combinations of negated/unnegated ψ_i. Then $\varphi \in j_{\mathcal{C}}(\Gamma)$, successively folding up (15). Thus we obtain $\varphi \in f(\Gamma)$, which is to say that $\Gamma \vdash_i \neg\neg\varphi$. ◀

For a related approach to Glivenko's theorem, using the *Jacobson radical* associated to a propositional theory, see [43]. In recent years, nuclei and radicals have received due attention in logic and proof theory. Cf., e.g., [38–42, 95].

Next we discuss how the folding property of radicals handles certain distributivity requirements arising in point-free topology.

▶ **Example 22.** Consider the co-coverage $\mathcal{C}$ of Example 12, generated by the axioms $(\bigwedge B, B)$, where B ranges over all subsets of L. The corresponding radical $j_{\mathcal{C}}$ renders the complete lattice $L_{j_{\mathcal{C}}}$ of fixed points of $j_{\mathcal{C}}$ into a *co-frame*. In fact, if $x \in L_{j_{\mathcal{C}}}$ and $Y \subseteq L_{j_{\mathcal{C}}}$, then

$$x \underset{j_{\mathcal{C}}}{\vee} (\bigwedge Y) = j_{\mathcal{C}}(x \vee \bigwedge Y) = \bigwedge_{y \in Y} j_{\mathcal{C}}(x \vee y) = \bigwedge_{y \in Y} x \underset{j_{\mathcal{C}}}{\vee} y,$$

simply because $x \vee Y \in \mathcal{C}(x \vee \bigwedge Y)$. Therefore, $L_{j_{\mathcal{C}}}$ satisfies the infinite distributive law characteristic for co-frames [71, II.1.2.2].

▶ **Example 23.** We continue the discussion of Example 10. Let R be a commutative ring with 1, and let L be the lattice of ideals $\mathfrak{a}$ of R. This L is multiplicative (Example 13), with

$$\mathfrak{a} \cdot \mathfrak{b} = \langle \{\, ab \mid a \in \mathfrak{a}, b \in \mathfrak{b} \,\} \rangle.$$

Moreover, $\langle 1 \rangle = R$ is a unit for this multiplication. It is well-known that L is a (unital commutative) *quantale* [78], for

$$\mathfrak{a} \cdot \bigvee_{i \in I} \mathfrak{b}_i = \bigvee_{i \in I} \mathfrak{a} \cdot \mathfrak{b}_i.$$

Consider the co-coverage $\mathcal{C}$ that is generated by the primality axioms $(\mathfrak{a}_1 \cdots \mathfrak{a}_k, \{\, \mathfrak{a}_1, \ldots, \mathfrak{a}_k \,\})$. The corresponding radical $j_{\mathcal{C}}$ transforms products to meets,

$$j_{\mathcal{C}}(\mathfrak{a} \cdot \mathfrak{b}) = j_{\mathcal{C}}(\mathfrak{a}) \wedge j_{\mathcal{C}}(\mathfrak{b}),$$

rendering $L_{j_{\mathcal{C}}}$ into a frame – it is, in fact, the frame of radical ideals of R [9]. Of course, the above construction makes sense for any unital commutative quantale Q whatsoever; the corresponding radical collapses Q to a frame.

4 Ideal elements

4.1 Spectra

Every co-coverage $\mathcal{C}$ specifies a set of "ideal" elements which split every covering instance. Provided that we now switch to a classical framework, these ideal elements form a basis of fixed elements for $j_{\mathcal{C}}$ (Proposition 26 and Proposition 31).

▶ **Definition 24.** *Let (a, B) be a nondeterministic axiom over L, and let $p \in L$. We write*

$$p \vDash (a, B) \quad \textit{for} \quad a \leqslant p \implies (\exists b \in B)\, b \leqslant p,$$

in which case we say that p splits *(a, B). If $\mathcal{E}$ is a set of nondeterministic axioms over L, then we write $p \vDash \mathcal{E}$ if p splits every member of $\mathcal{E}$. In particular, if $\mathcal{C}$ is a co-coverage on L, then we write*

$$p \vDash \mathcal{C} \quad \textit{for} \quad p \vDash \{\, (x, S) \mid x \in L \textit{ and } S \in \mathcal{C}(x) \,\}.$$

We denote the spectrum *of $\mathcal{C}$ as*

$$\mathrm{Spec}(\mathcal{C}) = \{\, p \in L \mid p \vDash \mathcal{C} \,\}.$$

Our choice of terminology is in line with the more prevalent use of the term "spectrum" as referring to the set of (proper) *prime elements* of L [48, V.4.1]; see Example 27(i) below for the corresponding co-coverage.

▶ **Lemma 25.** *If $\mathcal{E}$ is a set of nondeterministic axioms over L, then $p \vDash \mathcal{E}$ iff $p \vDash \mathcal{C}_{\mathcal{E}}$.*

Proof. Suppose that $p \vDash \mathcal{E}$. To see that $p \vDash \mathcal{C}_{\mathcal{E}}$, let $x \in L$ and $S \in \mathcal{C}_{\mathcal{E}}(x)$, and assume that $x \not\leqslant p$. There is some $(a, B) \in \mathcal{E}$ for which $S = B \vee x$ and $a \leqslant x$. Then $a \not\leqslant p$, and since p splits $\mathcal{E}$, there is $b \in B$ such that $b \not\leqslant p$. But then also $b \vee x \not\leqslant p$, and we have found some $s = b \vee x \in B \vee x$ for which $s \not\leqslant p$, as required.

Conversely, suppose that $p \vDash \mathcal{C}_{\mathcal{E}}$. Let $(a, B) \in \mathcal{E}$ and assume that $a \not\leqslant p$. Since $B \vee a \in \mathcal{C}_{\mathcal{E}}(a)$, there is some $b \in B$ such that $b \vee a \not\leqslant p$, hence also $b \not\leqslant p$. ◀

▶ **Proposition 26** (LEM)**.** *Let $\mathcal{C}$ be a co-coverage on L. If $p \in L$ splits $\mathcal{C}$, then $j_{\mathcal{C}}(p) = p$.*

Proof. Suppose that p splits $\mathcal{C}$. To establish that p is a fixed point of $j_{\mathcal{C}}$, it suffices by (12) to show, for every $x \in L$, that $x \sqsubseteq_{\mathcal{C}} p$ implies $x \leqslant p$. Consider to this end the complement $O_p = (\downarrow p)^c$. With LEM, it is straightforward to check that O_p is a Scott-open $\mathcal{C}$-distributor. Suppose now that $x \sqsubseteq_{\mathcal{C}} p$. Then, in particular, $x \sqsubseteq_{O_p} p$. Putting this contrapositively means that, for every $z \in L$, if $z \leqslant p$, then $x \vee z \leqslant p$. The case $z = 0$ yields the claim. ◀

In the proof of Proposition 26, notice that O_p is a $\mathcal{C}$-distributor precisely when p splits $\mathcal{C}$; it is only in this manner that the splitting property of ideal elements ensures stability with respect to the corresponding radical. Conversely, $j_{\mathcal{C}}(p) = p$ does *not* in general suffice for p to be prime. For instance, $j_{\mathcal{C}}(1) = 1$ always holds due to $j_{\mathcal{C}}$ being extensive, but 1 need not split $\mathcal{C}$ when the co-coverage requires ideal elements to be proper, i.e., strictly below 1.

▶ **Example 27.** Let's see how to capture some of the prime elements at work in domain theory, lattice theory, and point-free topology.

(i) An element p in a lattice L is said to be *prime* if $x \wedge y \leqslant p$ always implies that $x \leqslant p$ or $y \leqslant p$ [48, I.3.11.]. The prime elements are precisely those that split every instance of $(x \wedge y, \{ x, y \})$. The *proper* primes, i.e., those p such that $p \neq 1$, in addition split $(1, \emptyset)$. The *spectrum* [48, V.4.1] of a complete lattice is the set of all proper prime elements, which is precisely the spectrum (in the sense of this paper) of the co-coverage generated by the above axioms.

(ii) The (proper) *semiprime* elements of [71, V.4.] can be described by requiring that they split (next to properness) every instance of $(0, \{ x, y \})$ for those x, y with $x \wedge y = 0$. If instead we require that p be *a-semiprime* with respect to some $a \in L$, then we replace the former with $(0, \{ x_1, \dots, x_n \})$, while posing the side condition that $x_1 \wedge \cdots \wedge x_n \leqslant a$.

(iii) The *completely prime* [44] elements of a complete lattice L are those that split every instance of $(\bigwedge S, S)$ for $S \subseteq L$. This includes properness through the case of $S = \emptyset$.

(iv) The *co-atoms* [48, I.3.12.] (or *dual atoms* [51, I.6.3.]) of a lattice are the maximal elements < 1. With LEM it is not difficult to show that the co-atoms are precisely those $p \neq 1$ that split every axiom of the form $(0, \{ a \} \cup \{ b \mid a \vee b = 1 \})$. This is an instance of the *completeness* condition [67, Section 2], to which we next turn our attention.

4.2 From completeness to primality

Classical arguments based on maximality principles typically separate the existence of a maximal element from the verification that the latter satisfies a desired structural property, such as being prime. While the first step is handled by Zorn's lemma, the second step requires case-specific reasoning tailored to the situation at hand and makes essential use of classical logic.[6] Lemma 29 isolates this second step in an abstract form.

[6] Cf., e.g., the related discussion of [58, D4.5.14].

▶ **Definition 28.** *To every $O \subseteq L$, we associate a coverage $\mathcal{C}_O$, which is generated by the set of nondeterministic axioms of* O-completeness *[67, Section 2], i.e.,*

$$(a, \emptyset) \qquad [a \in O] \tag{16}$$

$$(0, \{\, a \,\} \cup (O : a)) \tag{17}$$

with the side condition on (16) *that $a \in O$, as indicated.*

For instance, the set of co-atoms (in the sense of Example 27(iv)) is precisely $\mathrm{Spec}(\mathcal{C}_{\{\,1\,\}})$.

▶ **Lemma 29** (LEM). *Let $\mathcal{C}$ be a co-coverage on L. If $O \in \mathrm{SOD}(\mathcal{C})$, then $\mathrm{Spec}(\mathcal{C}_O) \subseteq \mathrm{Spec}(\mathcal{C})$.*

Proof. Suppose that $O \in \mathrm{SOD}(\mathcal{C})$. Consider any $p \in \mathrm{Spec}(\mathcal{C}_O)$. By Lemma 25, this p splits the corresponding instances of (16) and (17). To see that $p \in \mathrm{Spec}(\mathcal{C})$, consider $x \in L$ and $S \in \mathcal{C}(x)$, and suppose that $x \leqslant p$. Working towards a contradiction, assume that $s \nleqslant p$ for every $s \in S$. Then, for every $s \in S$, since $p \vDash (0, \{\, s \,\} \cup (O : s))$, there must be some $b \leqslant p$ such that $b \vee s \in O$, which implies that $p \vee s \in O$, by upward closure. In all, $S \vee p \subseteq O$. However, by join-stability, $S \vee p \in \mathcal{C}(p)$, and since $O \in \mathrm{SOD}(\mathcal{C})$ it follows that $p \in O$. But then p would be required to split $(p, \emptyset)$, which is not possible. ◀

4.3 Prime radical

In [67] it has been shown that the proper O-complete elements form a basis for j_O:

▶ **Proposition 30** (AC). *For every Scott-open $O \subseteq L$, and every $x \in L$,*

$$j_O(x) = \bigwedge \mathrm{Spec}(\mathcal{C}_O)/x.$$

Proof. According to Lemma 25, the $p \in L$ which split $\mathcal{C}_O$ are precisely the *proper O-complete* elements of [67, (5) and 3.3], and hence the claim boils down to [67, Corollary 25]. ◀

Over **ZF**, the universal statement (for every complete lattice and Scott-open subset) of Proposition 30 is equivalent to AC [67, Proposition 28].

We can now state and prove our main *classical* result, the non-constructive counterpart to Proposition 16. This, too, is classically equivalent to AC (see Proposition 50 below).

▶ **Proposition 31** (AC). *Let $\mathcal{C}$ be a co-coverage on L. For every $x \in L$,*

$$j_{\mathcal{C}}(x) = \bigwedge \mathrm{Spec}(\mathcal{C})/x.$$

Moreover, for every $x \notin \bigcap \mathrm{SOD}(\mathcal{C})$, there is $p \in \mathrm{Spec}(\mathcal{C})$ such that $x \leqslant p$.

Proof. Combine (a) Proposition 26, (b) Lemma 29, (c) Proposition 30, and (d) Proposition 16(i), to see that

$$j_{\mathcal{C}}(x) \overset{(a)}{\leqslant} \bigwedge \mathrm{Spec}(\mathcal{C})/x \overset{(b)}{\leqslant} \bigwedge_{O \in \mathrm{SOD}(\mathcal{C})} \bigwedge \mathrm{Spec}(\mathcal{C}_O)/x \overset{(c)}{\leqslant} \bigwedge_{O \in \mathrm{SOD}(\mathcal{C})} j_O(x) \overset{(d)}{\leqslant} j_{\mathcal{C}}(x).$$

As regards the add-on, if $x \notin \bigcap \mathrm{SOD}(\mathcal{C})$, then $j_{\mathcal{C}}(x) \neq 1$ by Proposition 16(ii), so that $\mathrm{Spec}(\mathcal{C})/x$ cannot be empty, in view of the above. ◀

▶ **Corollary 32** (AC). *Every $f \in \mathrm{Clos}(L)$ is of the form $f = j_{\mathcal{C}}$ for a suitable co-coverage $\mathcal{C}$.*

Proof. Let $\mathcal{C} = \mathcal{C}_f$ (Definition 18). The fixed points of f are precisely the elements that split $\mathcal{C}_f$ so that, with Proposition 31, while keeping in mind the closure properties,

$$f(x) = \bigwedge \{ y \in L \mid x \leqslant y \text{ and } y = f(y) \} = \bigwedge \mathrm{Spec}(\mathcal{C}_f)/x = j_{\mathcal{C}}(x). \qquad \blacktriangleleft$$

▶ **Example 33** (AC). Following Stenström [93], the *radical* of a complete lattice L is defined as the meet of all dual atoms, and denoted $\mathrm{rad}(L)$. According to Example 27(iv),

$$\mathrm{rad}(L) = \bigwedge \mathrm{Spec}(\mathcal{C}_{\{1\}}).$$

Suppose now that L is compact, which is equivalent to the singleton $O = \{1\}$ being Scott-open. The corresponding case of Proposition 30 yields

$$\mathrm{rad}(L) = j_{\{1\}}(0),$$

which is to say that the radical of L is the join of small elements (cf. Example 4). – For algebraic lattices, this is [93, Theorem 8].

▶ **Example 34** (AC). We continue the discussion of Example 14, where L is the dual division lattice, and $\mathcal{C}$ is generated along (11). In this case, $\mathrm{Spec}(\mathcal{C}) = \mathbb{P}$, and the corresponding instance of Proposition 31 asserts that, for every positive integer n,

$$j_{\mathcal{C}}(n) = \mathrm{rad}(n) = \prod \{ p \in \mathbb{P} \mid p \mid n \}.$$

Hence we obtain the number-theoretic radical of n, simply defined as the product of prime divisors of n, prominently at work, e.g., in the ABC conjecture [50]. Because of $n \preccurlyeq j_{\mathcal{C}}(n)$, the prime factorization of n tumbles out as a consequence of AC via Theorem 31 through number theoretic descent. It is needless to say that invoking AC to this end is rather excessive.

▶ Remark 35. Recall from [48, I.3.8] that a subset $X \subseteq L$ is said to be *order generating* if every $x \in L$ can be written as $x = \bigwedge X/x$. Distributivity conditions are closely tied to the existence of order generating spectra. For instance, a continuous lattice is distributive iff the prime elements of Example 27(i) are order generating [48, I.3.14]. A similar role is played by co-atoms and completely prime elements vis-à-vis *complete distributivity*, see, e.g., [48, I.4.18] as well as [71, V.4] and [73]. It might be interesting to replace order generating sets systematically with the requirement that the corresponding radicals be identity mappings, but this goes beyond the scope of the present paper.

▶ Remark 36. In this section, we have tailored co-coverages $\mathcal{C}_O$ to subsets $O \subseteq L$. In the case in which O is Scott-open, it is natural to ask about the precise connection between the corresponding radicals. As it turns out,

$$O \in \mathrm{SOD}(\mathcal{C}_O) \quad \text{and (therefore)} \quad j_{\mathcal{C}_O} = j_O.$$

Let us prove this. We are working with a Scott-open set O anyway, so we concentrate on the distributor property with respect to $\mathcal{C}_O$. Thus, let $x \in L$ and $S \in \mathcal{C}_O(x)$, and suppose that $S \subseteq O$. We need to check that $x \in O$. According to the possible form of S (7), two cases need to be considered, which correspond to (16) and (17). In the first case (16) we have $S = \emptyset \vee x = \emptyset$ and some $a \in O$ with $a \leqslant x$, but then $x \in O$ is clear since O is upward closed. In the case of (17) there is $a \in L$ for which

$$S = (\{a\} \cup (O : a)) \vee x = \{ a \vee x \} \cup \{ b \vee x \mid b \in O : a \} \subseteq O. \tag{18}$$

But this set-inclusion means that (a) $a \vee x \in O$, as well as (b) for every $b \in L$, if $a \vee b \in O$ then $b \vee x \in O$. Combining (a) with (b) for the case in which $b = x$ immediately yields the required conclusion $x = b \vee x \in O$. Next, since $j_{\mathcal{C}_O}$ folds up $\mathcal{C}_O$, we know that, for every $x \in L$ and every such $S \in \mathcal{C}_O(x)$ of the latter form (18),

$$j_{\mathcal{C}_O}(x) = \bigwedge_{s \in S} j_{\mathcal{C}_O}(s) = j_{\mathcal{C}_O}(x \vee a) \wedge \bigwedge_{b \in O:a} j_{\mathcal{C}_O}(x \vee b).$$

To express it in terms of [67], $j_{\mathcal{C}_O}$ "folds up branchings" (5). In view of Proposition 2, it now suffices towards $j_{\mathcal{C}_O} = j_O$ that in addition $\nabla(j_{\mathcal{C}_O}) = O$. But axiom (16) ensures that every $\mathcal{C}_O$-distributor contains O, so that $O \subseteq \bigcap \mathrm{SOD}(\mathcal{C}_O)$, which is an equality due to $O \in \mathrm{SOD}(\mathcal{C}_O)$. Hence indeed $O = \nabla(j_{\mathcal{C}_O})$ by Proposition 16(ii).

4.4 Galois connection

Corollary 32 is part of an adjunction between co-coverages and closure operators on L, which is the aim of this brief detour, carried out in a classical framework, and mainly serving as a conceptual add-on.

▶ **Definition 37.** *Let* $\mathrm{CoCov}(L)$ *denote the collection of all co-coverages on* L*, partially ordered by stipulating*

$$\mathcal{C} \leqslant \mathcal{C}' \quad \textit{iff} \quad \mathrm{SOD}(\mathcal{C}') \subseteq \mathrm{SOD}(\mathcal{C}).$$

Notice that $\leqslant$ *is coarser than the pointwise inclusion order (Definition 5) between co-coverages: if* $\mathcal{C} \subseteq \mathcal{C}'$*, then* $\mathcal{C} \leqslant \mathcal{C}'$*.*

Recall that a *Galois connection* (see, e.g., [47, Def. 1.1.4] and [35]) between preordered sets $(P, \leqslant_P)$ and $(Q, \leqslant Q)$ is given by a pair of functions $f : P \to Q$ and $g : Q \to P$ such that, for every $p \in P$ and $q \in Q$,

$$f(p) \leqslant_Q q \quad \text{iff} \quad p \leqslant_P g(q), \tag{19}$$

in which case both f and g are monotone.

▶ **Proposition 38** (AC)**.** *There is a Galois connection*

$$\begin{tikzcd} \mathrm{Clos}\, L \arrow[r, bend left, "\mathcal{C}_\_"] & \mathrm{CoCov}\, L. \arrow[l, bend left, "j_\_"] \end{tikzcd}$$

Proof. On the one hand, if $\mathcal{C}_f \leqslant \mathcal{C}$, i.e., $\mathrm{SOD}(\mathcal{C}) \subseteq \mathrm{SOD}(\mathcal{C}_f)$, then Corollary 32 implies that

$$f(x) = j_{\mathcal{C}_f}(x) = \bigwedge_{O \in \mathrm{SOD}(\mathcal{C}_f)} j_O(x) \leqslant \bigwedge_{O \in \mathrm{SOD}(\mathcal{C})} j_O(x) = j_{\mathcal{C}}(x).$$

Conversely, if indeed $f \leqslant j_{\mathcal{C}}$, to check that $\mathcal{C}_f \leqslant \mathcal{C}$, let $O \in \mathrm{SOD}(\mathcal{C})$, consider $S \in \mathcal{C}_f(x)$, and suppose that $S \subseteq O$. We need to show that $x \in O$. This S is of the form $S = \{ f(a) \vee x \}$ for some $a \leqslant x$. Since $f(a) \vee x \leqslant j_{\mathcal{C}}(a) \vee x \leqslant j_{\mathcal{C}}(a \vee x)$ and since O is upward closed, we know that $j_{\mathcal{C}}(a \vee x) \in O$, and thus $x = a \vee x \in O$ because $j_{\mathcal{C}}$ is co-dense with respect to $\mathrm{SOD}(\mathcal{C})$. Thus $\mathrm{SOD}(\mathcal{C}) \subseteq \mathrm{SOD}(\mathcal{C}_f)$. ◀

Notice that AC has been used in the proof (through Proposition 31 and Corollary 32) only for the left-to-right implication of the respective instance of (19).

5 Applications

5.1 Finitary covering relations

Let S be a set. By a *covering relation* [23, 75, 79–81] on S we understand a relation $\lhd$ between elements a and subsets U of S which is *reflexive* (R) and *transitive* (T) in the following sense, written in rule notation:[7]

$$\frac{a \in U}{a \lhd U}\,(\mathrm{R}) \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}\,(\mathrm{T})$$

where $U \lhd V$ means that $b \lhd V$ for every $b \in U$. As a consequence, every covering relation is *monotone*, viz.,

$$\frac{a \lhd U}{a \lhd U, V}\,(\mathrm{M})$$

where we employ the usual shorthand notation to write U, V where it should actually read a set union $U \cup V$. A covering is said to be *finitary* if $a \lhd U$ implies that $a \lhd U_0$ for some $U_0 \in \mathrm{Fin}(U)$. As per usual, we write

$$U^{\lhd} = \{\, a \in S \mid a \lhd U \,\}$$

for the *saturation* of U; a subset $\mathfrak{a}$ of S is said to be *saturated* or an *ideal* (with respect to the covering at play) if $\mathfrak{a}^{\lhd} = \mathfrak{a}$, and we write

$$\mathrm{Sat}(\lhd)$$

for the set of all saturated subsets of S, ordered by set-inclusion. It is well-known that (finitary) covering relations $\lhd$ correspond with (algebraic) closure operators cl on $\mathrm{Pow}(S)$. This correspondence is set up as follows:

$$\begin{array}{lcl} \lhd & \rightsquigarrow & \mathrm{cl}_{\lhd} : U \mapsto U^{\lhd}, \\ \mathrm{cl} & \rightsquigarrow & a \lhd_{\mathrm{cl}} U \equiv a \in \mathrm{cl}(U). \end{array}$$

In particular, $\mathrm{Sat}(\lhd) \subseteq \mathrm{Pow}(S)$ is the complete lattice of fixed points of the associated closure operator, with bottom element $\emptyset^{\lhd} = \{\, a \in S \mid a \lhd \emptyset \,\}$ and top element $S = S^{\lhd}$, and where the meet and join of a family $\mathfrak{a}_i$ $(i \in I)$ of ideals is given, respectively, by

$$\bigwedge_{i \in I} \mathfrak{a}_i = \bigcap_{i \in I} \mathfrak{a}_i \quad \text{and} \quad \bigvee_{i \in I} \mathfrak{a}_i = \left(\bigcup_{i \in I} \mathfrak{a}_i \right)^{\lhd}.$$

▶ **Definition 39.** *Let $\lhd$ be a covering on S. By a* geometric axiom [59] *we understand a pair*

$$(A, \mathcal{B}) \in \mathrm{Fin}(S) \times \mathrm{Pow}(\mathrm{Fin}(S)).$$

If $\mathcal{G}$ is a set of geometric axioms, let $\lhd_{\mathcal{G}}$ denote the extension of $\lhd$ that is inductively generated through $\lhd$ and the rules $(\mathrm{T}_{(A,\mathcal{B})})$ that correspond to the geometric axioms in $\mathcal{G}$, viz.,[8]

$$\frac{a \lhd U}{a \lhd_{\mathcal{G}} U}\,(\mathrm{E}) \quad \textit{and} \quad \frac{(\forall B \in \mathcal{B})\, a \lhd_{\mathcal{G}} U, B}{a \lhd_{\mathcal{G}} U, A}\,(\mathrm{T}_{(A,\mathcal{B})}).$$

---

[7] There is a certain foundational mismatch in that covering relations as we understand them here more often than not are placed within a predicative setting; see, e.g., [23, 79–81].

[8] Inductive generation is to be understood in the generalized sense of [1].

We say that $\lhd_{\mathcal{G}}$ *is* conservative *over* $\lhd$ *if, for every* $U \in \mathrm{Fin}(S)$ *and* $a \in S$*,*

$$a \lhd_{\mathcal{G}} U \implies a \lhd U,$$

which by (E) *is the case precisely when* $\lhd$ *and* $\lhd_{\mathcal{G}}$ *coincide.*

The extension $\lhd_{\mathcal{G}}$ may be a covering relation only in specific cases. However, we will further extend $\lhd_{\mathcal{G}}$ through a radical, which anyway corresponds to a (not necessarily finitary) covering relation.

▶ **Definition 40.** *Let* $\lhd$ *be a covering on* S*, and let* $\mathcal{G}$ *be a set of geometric axioms. Let* $\mathcal{C} = \mathcal{C}_{\mathcal{E}_{\mathcal{G}}}$ *be the co-coverage on* $L = \mathrm{Sat}(\lhd)$ *that is generated by*

$$\mathcal{E}_{\mathcal{G}} = \{\, (A^{\lhd}, \{\, B^{\lhd} \mid B \in \mathcal{B} \,\}) \mid (A, \mathcal{B}) \in \mathcal{G} \,\}\,.$$

Because $U \subseteq \mathfrak{a}$ *iff* $U^{\lhd} \subseteq \mathfrak{a}$ *for ideals* $\mathfrak{a}$*, it follows immediately that* $\mathfrak{p} \in \mathrm{Spec}\,\mathcal{C}$ *iff*

$$(\forall (A,\mathcal{B}) \in \mathcal{G})(\, A \subseteq \mathfrak{p} \implies (\exists B \in \mathcal{B}) B \subseteq \mathfrak{p}\,).$$

▶ **Proposition 41.** *Let* $\lhd$ *be a finitary covering on* S*, and let* $\mathcal{G}$ *be a set of geometric axioms. Let* $\mathcal{C} = \mathcal{C}_{\mathcal{E}_{\mathcal{G}}}$ *denote the generated co-coverage on* $\mathrm{Sat}(\lhd)$*, and let* $j = j_{\mathcal{C}}$ *denote the corresponding radical. The following are equivalent.*

(i) $j(\mathfrak{a}) = \mathfrak{a}$ *for every* $\mathfrak{a} \in \mathrm{Sat}(\lhd)$*.*
(ii) $\lhd_{\mathcal{G}}$ *is conservative over* $\lhd$*.*
(iii) $\uparrow\!(x^{\lhd}) \in \mathrm{SOD}(\mathcal{C})$ *for every* $x \in S$*.*

Proof. (i) ⇒ (ii) Suppose that the ideals of $\lhd$ are fixed under j. We argue by induction to see that $\lhd_{\mathcal{G}}$ is conservative over $\lhd$. This is trivial for (E). As for applications of $(\mathrm{T}_{(A,\mathcal{B})})$, suppose by induction that $a \lhd U, B$ for every $B \in \mathcal{B}$. By the construction of $\mathcal{C}$ (Lemma 7) we have $\{\, B^{\lhd} \vee A^{\lhd} \mid B \in \mathcal{B} \,\} \in \mathcal{C}(A^{\lhd})$, so that

$$\{\, U^{\lhd} \vee B^{\lhd} \vee A^{\lhd} \mid B \in \mathcal{B} \,\} \in \mathcal{C}(U^{\lhd} \vee A^{\lhd})$$

by join-stability (Definition 5(ii)). Since j is extensive and folds up $\mathcal{C}$, we obtain

$$\begin{aligned} a \in \bigcap_{B \in \mathcal{B}} (U \cup B)^{\lhd} \subseteq \bigcap_{B \in \mathcal{B}} j\left((U \cup B \cup A)^{\lhd}\right) &= \bigcap_{B \in \mathcal{B}} j\left(U^{\lhd} \vee B^{\lhd} \vee A^{\lhd}\right) \\ &= j\left(U^{\lhd} \vee A^{\lhd}\right) = (U \cup A)^{\lhd} \end{aligned}$$

from the assumption that the ideals are fixed under j, hence indeed $a \lhd U, A$.

(ii) ⇒ (iii) Let $x \in S$ and consider

$$D_x := \uparrow\!(x^{\lhd}) = \{\, \mathfrak{a} \in \mathrm{Sat}(\lhd) \mid x \in \mathfrak{a} \,\}\,,$$

which is Scott-open, because $\lhd$ is finitary [75, Proposition 7]. To check that D_x is a distributor for the co-coverage at hand, consider $\mathfrak{x} \in \mathrm{Sat}(\lhd)$ along with $\mathcal{S} \in \mathcal{C}(\mathfrak{x})$, and suppose that $\mathcal{S} \subseteq D_x$. This $\mathcal{S}$ is of the form $\mathcal{B} \vee \mathfrak{x} = \{\, B^{\lhd} \vee \mathfrak{x} \mid B \in \mathcal{B} \,\}$ for some $(A, \mathcal{B}) \in \mathcal{G}$ with $A^{\lhd} \subseteq \mathfrak{x}$, and we need to show that $\mathfrak{x} \in D_x$. The assumption tells us that $x \lhd \mathfrak{x}, B$ for every $B \in \mathcal{B}$. Through (E) and $(\mathrm{T}_{(A,\mathcal{B})})$ we obtain $x \lhd_{\mathcal{G}} \mathfrak{x}, A$, hence $x \lhd \mathfrak{x}, A$ because $\lhd_{\mathcal{G}}$ is conservative over $\lhd$. Since $A \subseteq A^{\lhd} \subseteq \mathfrak{x}$, we see that $x \lhd \mathfrak{x}$, as was to be shown.

(iii) ⇒ (i) Suppose that every such D_x as displayed above is a $\mathcal{C}$-distributor. Consider any $\mathfrak{a} \in \mathrm{Sat}(\lhd)$. To see that $j(\mathfrak{a}) = \mathfrak{a}$, by (12) it suffices to show that if $\mathfrak{b} \sqsubseteq_{\mathcal{C}} \mathfrak{a}$, then $\mathfrak{b} \subseteq \mathfrak{a}$. Accordingly, consider any $\mathfrak{b} \in \mathrm{Sat}(\lhd)$ and suppose that $\mathfrak{b} \sqsubseteq_{\mathcal{C}} \mathfrak{a}$. Let $x \in \mathfrak{b}$. Since $D_x \in \mathrm{SOD}(\mathcal{C})$, we know that $\mathfrak{b} \sqsubseteq_{D_x} \mathfrak{a}$ in particular. Clearly $\mathfrak{b} = \mathfrak{b} \vee \emptyset^{\lhd} \in D_x$, which now implies $\mathfrak{a} = \mathfrak{a} \vee \emptyset^{\lhd} \in D_x$, i.e., $x \in \mathfrak{a}$. ◀

▶ **Example 42.** Let S be a set with a finitary covering relation $\lhd$, and suppose that there is a binary mapping $\circ$ on S such that, for every $a, b, c \in S$ and $U \subseteq S$,

$$\frac{c \lhd U, a \qquad c \lhd U, b}{c \lhd U, a \circ b}$$

This pivotal condition is coined *Encoding* in [75]. In this setting, an ideal $\mathfrak{a}$ is said to be *prime* if the membership $a \circ b \in \mathfrak{a}$ always implies that $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$. The prime ideals are precisely the ideals that split every axiom of the form $(\{ a \circ b \}^{\lhd}, \{ \{ a \}^{\lhd}, \{ b \}^{\lhd} \})$. Under the assumption that Encoding holds, the corresponding geometric extension is conservative. According to Proposition 41, every ideal is closed with respect to the corresponding radical. Let us now assume AC: Proposition 31 implies that every ideal $\mathfrak{a}$ is the intersection of all prime ideals which contain $\mathfrak{a}$. – This is the *Universal Krull–Lindenbaum Theorem* [75, Theorem 14].

5.2 Scott's entailment relations

Example 42 is but one instance of a more general phenomenon that arises between single-conclusion entailment relations and their multi-conclusion counterparts [76]. Let us briefly consider the latter. Let S be a set. A Scott-style[9] (multi-conclusion) *entailment relation* [21, 88, 90] on S is a relation $\vdash$ between finite subsets V, W of S which is reflexive (R), monotone (M), and transitive (T), in the following sense, written again in rule notation:

$$\frac{V \between W}{V \vdash W}\,(\mathrm{R}) \qquad \frac{V \vdash W}{V, V' \vdash W, W'}\,(\mathrm{M}) \qquad \frac{V \vdash W, a \qquad V, a \vdash W}{V \vdash W}\,(\mathrm{T}).$$

Transitivity is a form of Gentzen's *cut* rule. Again we employ the common shorthand notation V, a to abbreviate the set union $V \cup \{ a \}$, etc. To every such $\vdash$ we can associate a canonical covering relation: the *trace* $\blacktriangleleft$ of $\vdash$ is given by all the single conclusion instances of $\vdash$, which by default is a finitary covering:

$$a \blacktriangleleft U \;\equiv\; (\exists U_0 \in \mathrm{Fin}(U))\, U_0 \vdash a.$$

Suppose now that we have at hand some finitary covering relation $\lhd$ on S, of which the trace $\blacktriangleleft$ of $\vdash$ is an extension, i.e., such that, for every $a \in S$ and $U \in \mathrm{Pow}(S)$,

$$a \lhd U \implies a \blacktriangleleft U, \tag{20}$$

in which case $\mathrm{Sat}(\blacktriangleleft) \subseteq \mathrm{Sat}(\lhd)$. This situation (20) may arise from considering multi-conclusion axioms on top of some $\lhd$ [76], and calls for an answer as to whether $\blacktriangleleft$ is conservative over $\lhd$, meaning that (20) is an equivalence [76, p. 230]. To address this, we work with $L = \mathrm{Sat}(\lhd)$ and consider the set $\mathcal{G}$ of finitary geometric axioms

$$(A, \{ \{ b \} \mid b \in B \}) \quad \text{for every } A, B \in \mathrm{Fin}(S) \text{ with } A \vdash B. \tag{21}$$

It is not difficult to check that in the present setting $\lhd_{\mathcal{G}} = \blacktriangleleft$, and that the conservation criterion of Proposition 41 translates into the requirement that

$$\frac{A \vdash B \qquad (\forall b \in B)\, c \lhd U, b}{c \lhd U, A} \tag{22}$$

holds, which boils down to the case of generating entailments $A \vdash B$ whenever $\vdash$ is inductively generated.[10] – This is [76, Theorem 2].

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[9] See [27, 28, 69] for a discussion of Lorenzen's precedence [64].

[10] I.e., inductively generated from a distinguished set of initial entailments [77], given at the outset, rather than $\vdash$ itself.

As far as semantics is concerned (now assuming AC), by a *model* of $\vdash$ we understand a subset $\mathfrak{p} \subseteq S$ such that $\mathfrak{p} \between W$ whenever $\mathfrak{p} \supseteq V$ and $V \vdash W$. We denote the set of all models of $\vdash$ with $\mathrm{Mod}(\vdash)$. Alternatively, the models can now be described by means of the present terminology: if $\mathcal{C}$ denotes the co-coverage generated by $\mathcal{G}$ according to Definition 40, then $\mathrm{Mod}(\vdash) = \mathrm{Spec}(\mathcal{C})$. By the corresponding instance of Proposition 31, combined with Proposition 41, conservativity of $\vdash$ over $\lhd$ is equivalent to the requirement that every saturated set (of $\lhd$) be an intersection of models of $\vdash$. – This is [76, Theorem 6].

5.3 Henkin radical

Consider a signature Σ of first-order logic, and let $\Gamma \cup \{ \varphi \}$ be a set of formulas over Σ. We consider classical derivability, and write $\Gamma \vdash \varphi$ if there is a (natural) derivation with conclusion φ and with every (uncancelled) hypothesis in Γ. Recall (e.g., [92, 8.6.5. Definition]) that a set Γ of formulas over Σ is said to be *prime* if the following conditions hold for every formula φ, ψ over Σ:

(i) if $\Gamma \vdash \varphi \vee \psi$, then either $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$,
(ii) if $\Gamma \vdash \exists x \varphi(x)$, then $\Gamma \vdash \varphi(c)$ for some constant $c \in \Sigma$.

Working towards a radical-theoretic version of Henkin's lemma on the existence of prime extensions, we first need to capture the primality condition in terms of a splitting property. To this end, we consider a set K of new constants c_φ, one for every Σ-formula of the form $\exists x \varphi(x)$. Now let S be the set of formulas over $\Sigma \cup K$, and consider the finitary covering $\lhd$ on S as given by derivability in classical first-order logic, viz.,

$$\varphi \lhd \Gamma \equiv \Gamma \vdash \varphi.$$

Let $L = \mathrm{Sat}(\lhd)$ be the corresponding complete lattice of deductively closed sets of formulas. Last but not least, let the co-coverage $\mathcal{C}$ be generated by the following geometric axioms which correspond, respectively, to the primality conditions recalled above:

$$(\{ \varphi \vee \psi \}^{\lhd}, \{ \{ \varphi \}^{\lhd}, \{ \psi \}^{\lhd} \}) \tag{23}$$
$$(\{ \exists x \varphi \}^{\lhd}, \{ \{ \varphi[c/x] \}^{\lhd} \mid c \in K \}) \tag{24}$$

▶ **Proposition 43.** *If Γ is a deductively closed set of formulas over Σ, then $j_{\mathcal{C}}(\Gamma) = \Gamma$.*

Proof. According to Proposition 41, it suffices to show that $D_\chi = \{ \Gamma \mid \Gamma \vdash \chi \} \in \mathrm{SOD}(\mathcal{C})$ for every formula χ over Σ. Since $\lhd$ is finitary, this D_χ is Scott-open [75, Proposition 7], so that we may concentrate on the distributor property, which boils down to the rules ($\vee$E) and ($\exists$E). In fact, consider some theory Γ, let $\mathcal{S} \in \mathcal{C}(\Gamma)$, and suppose that $\mathcal{S} \subseteq D_\chi$. By (7), this $\mathcal{S}$ is of the form $\mathcal{S} = \mathcal{B} \vee \Gamma$, where $\mathcal{B}$, according to (23) and (24), is either of the form

(a) $\mathcal{B} = \{ \{ \varphi \}^{\lhd}, \{ \psi \}^{\lhd} \}$ with $\varphi \vee \psi \in \Gamma$; or
(b) $\mathcal{B} = \{ \{ \varphi(c) \}^{\lhd} \mid c \in K \}$ with $\exists x \varphi \in \Gamma$.

In the first case (a), we know that $\Gamma, \varphi \vdash \chi$ as well as $\Gamma, \psi \vdash \chi$, whence $\Gamma, \varphi \vee \psi \vdash \chi$ by ($\vee$E), which is to say that $\Gamma = \Gamma \vee \{ \varphi \vee \psi \}^{\lhd} \in D_\chi$. The second case (b) translates to $\Gamma, \varphi(c) \vdash \chi$ for every $c \in K$. In particular, $\Gamma, \varphi(c_\varphi) \vdash \chi$ for the new constant $c_\varphi \in K$. Now ($\exists$E) allows to infer $\Gamma, \exists x \varphi(x) \vdash \chi$, hence again $\Gamma = \Gamma \vee \{ \exists x \varphi \}^{\lhd} \in D_\chi$. ◀

▶ Remark 44 (AC). If $\Gamma \nvdash \varphi$, then, by Proposition 43 along with Proposition 31, there is a theory $\Gamma' \supseteq \Gamma$ over $\Sigma \cup K$ which is prime and such that $\Gamma' \nvdash \varphi$ (cf. [92, 8.6.6. Lemma]). – This is Henkin's lemma [52] as a consequence of the spectral representation of $j_{\mathcal{C}}$.

5.4 Resolving nondeterminacy

Let again L be a complete lattice, the compact elements of which now represent finite information states. Following [98], by a *nondeterministic computation* in L we understand a finitely branching rooted tree $(T, \prec)$, together with a monotone mapping

$$\mathrm{val} : T \to KL,$$

assigning to each node of T the information accumulated at that stage of the computation. Monotonicity ensures that information grows along the paths of computations. We suppose that T is discrete, i.e., that the equality relation is decidable, and that it is decidable whether any given node is a leaf. Here, by a *path* through T we understand a finite or infinite sequence

$$(t_n)_{n \leqslant k} \quad \text{or} \quad (t_n)_{n \in \mathbb{N}}$$

of nodes such that t_0 is the root of T, and such that t_n is the parent of t_{n+1}. For a node t of T, by the *depth* of t we understand the length of the unique path from the root to t.[11] With an eye towards Proposition 47 below, we further borrow jargon from intuitionistic logic [72]:

▶ **Definition 45.** *A* bar *in the computation tree T is a set $\beta \subseteq T$ of nodes such that every branch of T meets β, i.e., for every finite maximal or infinite path (t_n) starting at the root, there exists n with $t_n \in \beta$. A bar β has* uniform depth N *if every branch meets β at some node the depth of which is at most N.*

Next, let $\mathcal{E}_T$ consist of all branchings, i.e., we consider the nondeterministic axioms

$$(\mathrm{val}(t), \{\, \mathrm{val}(t') \mid t \prec t' \,\}) \tag{25}$$

where t' is supposed to range over the children of t only. Let $\mathcal{C} = \mathcal{C}_{\mathcal{E}_T}$ denote the generated co-coverage, and let $j = j_{\mathcal{C}}$ denote the corresponding radical. The ideal elements of $\mathcal{C}$ can be understood as representing outcomes of the computation, corresponding to (potentially) infinite paths through the tree. Indeed, splitting the co-coverage requires that, at each node, one successor (if available) be selected, thereby determining a path through T.

▶ **Definition 46.** *We introduce a relation $\vDash \subseteq T \times \mathrm{Pow}(L)$ by stipulating*

$$t \vDash U \quad \textit{for} \quad (\exists x \in U)\, x \leqslant j(\mathrm{val}(t)),$$

in which case we say that t satisfies *U. By monotonicity of* val *and j, once t satisfies U, so do the children of t.*

▶ **Proposition 47.** *Let $O \in \mathrm{SOD}(\mathcal{C})$. If $\beta \subseteq T$ is a bar of uniform depth, and every node $t \in \beta$ satisfies O, then so does the root of T.*

Proof. We argue by induction on the depth N, the base case for which is trivial. Thus, consider a bar β of uniform depth $N+1$, every node of which satisfies P. We show that this folds up to a bar β' of depth N. Start with β' containing all leaves at depths strictly less than N. Next go through the nodes t at depth N and distinguish cases: (a) if any such t is a leaf, then $t \in \beta$ anyway, so that $t \vDash O$ by assumption, and we add t to β'; (b) if instead t is a parent, then we consider the set $\{\, t_1, \dots, t_k \,\} \subseteq \beta$ of t's children. By assumption on β, the t_i's satisfy O, which is to say that there are $x_1, \dots, x_k \in O$ such that

$$x_1 \leqslant j(\mathrm{val}(t_1)) \quad \dots \quad x_k \leqslant j(\mathrm{val}(t_k)). \tag{26}$$

[11] This notion of depth applies only to accessible nodes, i.e., those lying on a path.

Since j is co-dense with respect to O, this implies that $\mathrm{val}(t_1), \ldots, \mathrm{val}(t_k) \in O$, hence $\mathrm{val}(t) \in O$ by the distributor property. In particular, $t \vDash O$, and we add t to β'. Proceeding in this manner, the resulting set β' is a bar at depth N, and we may invoke the induction hypothesis towards the conclusion. ◀

▶ **Example 48.** Consider again a finitary covering relation $\lhd$ on a set S. Following [84], let us consider a set $\mathcal{E}$ of nondeterministic axioms of the form

$$(A, B) \in \mathrm{Fin}(S) \times \mathrm{Fin}(S).$$

A subset $R \subseteq S$ is said to be *regular* [84, 2.3] if, for every $(A, B) \in \mathcal{E}$ and $U \in \mathrm{Fin}(S)$,

$$\frac{(\forall b \in B)\,(U, b)^{\lhd} \between R}{(U, A)^{\lhd} \between R}.$$

We consider computations in $\mathrm{Sat}(\lhd)$. Branchings may occur at nodes labelled with a finitely generated ideal $\mathfrak{a} = U^{\lhd}$ whenever we find that $A \subseteq \mathfrak{a}$ for some $(A, B) \in \mathcal{E}$; the children are each labelled $(U, b)^{\lhd}$, one for each $b \in B$. According to [84, Proposition 4.6], if any such computation *terminates* in a regular set R, i.e., if the ideals sitting at the leaves meet R, then so does the root. – This becomes a ready instance of Proposition 47. In fact, if R is regular, then $O_R = \{\, \mathfrak{a} \in \mathrm{Sat}(\lhd) \mid \mathfrak{a} \between R \,\}$ is a Scott-open distributor for the related co-coverage (25), and the desired conclusion follows by translating the assumptions into the present setting.

▶ Remark 49. There are possible variations of Proposition 47. For instance, the statement carries over to every *meet-closed* $M \subseteq L$ in place of the Scott-open $\mathcal{C}$-distributor O. The inductive step (26) in the proof of Proposition 47 is then a matter of folding up via j,

$$x_0 := x_1 \wedge \cdots \wedge x_k \leqslant \bigwedge_{i=1}^{k} j(\mathrm{val}(t_i)) = j(\mathrm{val}(t)) \in M,$$

and the remainder of the proof goes through unmodified.

6 Some reverse perspectives

A careful distinction has been maintained between constructive results and the conclusions that can be drawn within a classical framework. In the sense of Coquand and Lombardi's notion of a constructive version of an abstract algebraic theorem – one having "a clear computational content" and yielding the classical statement upon "application of a well classified non-constructive principle" [25] – Proposition 31 indeed represents the classical counterpart to Proposition 16. In this section, we show that there is little leeway in generality – the full strength of AC has been coded into Proposition 31.

Let us write IP (Intersection Principle) for the *universal* statement of Proposition 31, i.e., for the claim that, for every complete lattice L, every $\mathcal{C} \in \mathrm{CoCov}(L)$, and every $x \in L$, we have that

$$j_{\mathcal{C}}(x) = \bigwedge \mathrm{Spec}(\mathcal{C})/x. \tag{IP}$$

To establish that IP implies AC, we go through the universal statement (for every complete lattice and every Scott-open subset) of Proposition 30, which by contraposition yields Krull's maximal ideal theorem [67, Example 27], a prominent **ZF**-equivalent of AC [8, 53, 54, 61].

▶ **Proposition 50** (LEM)**.** IP *implies* AC.

Proof. To gain back Proposition 30 from IP, combine the observation of Remark 36 that $j_O = j_{\mathcal{C}_O}$ with IP for the particular case of $\mathcal{C}_O$ to see that $j_O(x) = \bigwedge \mathrm{Spec}(\mathcal{C}_O)/x$. ◀

It is a time-honored result of constructive and intuitionistic set theory that intuitionistic logic is incompatible with AC [2, 14, 32, 34, 49]. Zorn's lemma, on the other hand, is said to be *constructively neutral* [12]. However, mere maximality is too "anonymous" to exhibit, say, a primality property. We have used LEM towards Proposition 31. Classical logic has in fact been properly coded into this result:[12]

▶ **Proposition 51.** IP *implies* LEM.

Proof. We go through Bell's observation that the following statement, quoted verbatim from [13, p. 161], implies LEM in **IZF**:[13]

In each Boolean algebra the intersection of the family of all its prime filters is $\{1\}$.

To show that this statement is indeed a consequence of IP, we tie in with the discussion of Section 5.2. Let B be a Boolean algebra, and define $\lhd$ and $\vdash$ by stipulating

$$a \lhd U \quad \text{iff} \quad (\exists U_0 \in \mathrm{Fin}(U)) \bigwedge U_0 \leqslant a \qquad \text{and} \qquad V \vdash W \quad \text{iff} \quad \bigwedge V \leqslant \bigvee W$$

for $a \in B$ and $U \in \mathrm{Pow}(B)$, and $V, W \in \mathrm{Fin}(B)$. The conservation criterion (22) is an easy consequence of the distributive law in B. According to Proposition 41, the radical on $\mathrm{Sat}(\lhd)$ associated with the finitary geometric axioms of $\vdash$ is the identity mapping. The corresponding ideal elements are precisely the prime filters of B, and the spectral representation via IP yields the desired statement quoted above. Towards LEM, proceed along [13, p. 161 sq.]. ◀

In light of Proposition 51, we can now improve Proposition 50 to an equivalence between IP and AC over **IZF**.[14]

▶ **Corollary 52.** IP *is equivalent to* AC. ◀

Let us conclude with a question concerning foundations. To what extent do the results communicated in this note admit predicative reformulations, for example in **CZF**? There are indications that this may be possible in specific cases, e.g., by restricting attention to set-generated lattices such as the algebraic ones arising in ring theory.[15] For instance, [67, Prop. 15] shows that membership in j_O can be characterized by quantification over KL alone:

$$x \leqslant j_O(y) \quad \text{iff} \quad (\forall b \in KL)(\, x \vee b \in O \to y \vee b \in O\,),$$

which is an improvement insofar as a complete lattice cannot be proved in **CZF** to be carried by a set [33, Theorem 2.5], yet with a generating set at hand, size issues might be kept at bay. By contrast, the proper class of Scott-open distributors appearing in the definition of $j_{\mathcal{C}}$ seems to present a genuine obstacle at present.

[12] Completeness principles have caught attention also from constructive reverse mathematics, cf., e.g., [89].
[13] This proved convenient also towards [76, Proposition 4] and [84, Proposition 6.6].
[14] One of the anonymous reviewers kindly pointed out this improvement, along with several simplifications in Section 6 of this paper.
[15] In this vein, one of the anonymous reviewers posed the challenge of translating the applications of Section 5 into a predicative setting.

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