Model Checking for Low Monodimensionality Fragments of CMSO on Topological-Minor-Free Graph Classes

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Abstract

Algorithmic meta-theorems explain the tractability of large classes of computational problems by linking logical expressibility with structural graph properties. While extensions of first-order logic such as FO+dp admit efficient model checking on graph classes excluding a fixed topological minor, comparable results for richer fragments of CMSO were previously unknown. We further develop the framework of Sau, Stamoulis, and Thilikos [SODA 2025] for fragmenting CMSO via *annotated graph parameters*, which restrict set quantification to vertex sets satisfying bounded structural conditions. Following this approach, we identify a fragment of CMSO, namely the one defined by allowing quantification only over sets having what we call *low monodimensionality*, that generalizes several previously-known logics and we show that model checking for this fragment, enhanced with the disjoint-paths predicate, is fixed-parameter tractable on topological-minor-free graph classes. Such classes essentially delimit the tractability for this logic on subgraph-closed classes. As a consequence, our results lift several known algorithmic meta-theorems beyond first-order logic to the topological-minor-free setting.

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1 Introduction

Algorithmic meta-theorems. Algorithmic meta-theorems (AMTs) aim to explain, in a uniform and conceptually clean way, *why* large families of computational problems admit efficient algorithms on large classes of inputs. Rather than designing a new algorithm for each individual problem and each individual graph class, AMTs identify general principles showing that *all* problems expressible in a certain logic can be solved efficiently on *all* graphs from a certain structural class. In this sense, meta-theorems do not merely provide algorithms; they unify, systematize, and conceptually explain tractability.

Formally, an AMT typically asserts that for a logic $\mathcal{L}$ and a graph class $\mathscr{C}$, the model checking problem, that is, deciding whether a given graph $G \in \mathscr{C}$ satisfies a given formula $\varphi \in \mathcal{L}$, in symbols $G \models \varphi$, can be solved in time

$$f(|\varphi|, \mathscr{C}) \cdot |G|^c,$$

for some function f, where the exponent c is independent of both the formula φ and the class $\mathscr{C}$. Such results show that once the structural complexity of the input and the expressive power of the specification language are bounded, the combinatorial explosion inherent in many problems can be controlled. Over the past decades, AMTs have become a central paradigm in algorithmic graph theory and finite model theory, unifying a wide range of algorithmic techniques across many problems and graph classes [36, 38, 43, 60].

Monadic second-order logic with modulo counting. The classical and most influential example of an algorithmic meta-theorem is Courcelle's theorem [13–15]. It states that every graph property definable in monadic second-order logic with modulo counting (CMSO) can be decided in linear time on classes of graphs of bounded treewidth. This theorem abstracts dynamic programming over tree decompositions and provides a single, uniform explanation for the tractability of a vast number of problems on tree-like graph classes. It has been extended to related width parameters, most notably cliquewidth [17]. At the same time, matching lower bounds show that these results are essentially tight. Indeed, extending efficient CMSO model checking to substantially more general graph classes is not to be expected [33, 44].

First-order logic. A different and highly successful line of research focuses on first-order logic (FO), which is significantly less expressive than CMSO, but enjoys much broader algorithmic tractability. Model checking for FO is tractable on a wide range of sparse graph classes, including bounded degree graphs [59], bounded expansion and nowhere dense classes [27, 39]. These algorithms rely on locality principles and structural decompositions [19, 31], and are optimal for subgraph-closed classes, that is, FO model checking on a subgraph-closed class is fixed-parameter tractable if and only if the class is nowhere dense [26]. Recently, the tractability results have also been extended to dense graph classes, e.g. to classes of bounded twin-width [9]. A main conjecture from the area states that FO model checking is fixed-parameter tractable on a class of graphs closed under induced subgraphs if and only if the class is monadically dependent, see [8], and partial results as well as a beautiful structure theory for monadically dependent classes is being developed, see e.g. [11, 21–24, 32].

From a high-level perspective, these results reveal a clear tradeoff: FO is tractable on very broad graph classes, but its expressive power is limited. In particular, FO cannot express unbounded counting properties, reachability, or global connectivity. Thus, many fundamental algorithmic problems lie beyond its expressive scope.

The gap between FO and CMSO. For a long time, there was a striking gap between the two extremes: on the one hand, CMSO, which is very expressive but only tractable on rather restricted graph classes, and on the other hand, FO, which is widely tractable but too weak to express many important properties. It was natural to ask whether there exist logics in between these two that combine *meaningful expressive power* with *broad algorithmic tractability*.

Classical candidates for closing this gap include (monadic) transitive closure logic, (monadic) least fixed-point logic, and related extensions of first-order logic, see [28, 47]. However, these logics turn out to be intractable already on very simple graph classes, such as planar graphs of maximum degree three. As a consequence, their study did not lead to robust algorithmic meta-theorems, and they failed to provide a satisfying explanation of tractability beyond very restricted settings [36]. Other extensions are those with counting or other numerical predicates, see e.g. [25, 40, 45, 46].

New algorithmically meaningful extensions of FO. Recently, a new and promising direction emerged. Instead of adding general recursion mechanisms, recent work has begun to directly enrich first-order logic with carefully chosen operators that are strong enough to express important algorithmic properties, yet sufficiently tame to preserve tractability on broad graph classes. The guiding principle is to extend FO in a way that aligns with known algorithmic techniques and structural decompositions, thereby enabling genuine algorithmic meta-theorems that go beyond pure first-order expressiveness.

One approach enhances FO with restricted connectivity predicates, leading to logics such as FO+conn, also called *separator logic* [7, 58]. Model checking for FO+conn-expressible formulas can be performed in time $f(|\varphi|, |H|) \cdot |G|^3$ on H-topological-minor-free graphs [49]. A stronger extension, FO+dp [58], allows predicates expressing the existence of vertex-disjoint paths. More precisely, FO is enhanced with the $2k$-ary *disjoint-paths predicate* $\mathsf{dp}_k(x_1, y_1, \ldots, x_k, y_k)$, where $x_1, y_1, \ldots, x_k, y_k$ are first-order terms, which expresses the existence of k pairwise vertex-disjoint paths between the valuations of x_i and y_i, for $i \in \{1, \ldots, k\}$. With the help of the disjoint-paths predicate, FO+dp can define global properties, including connectivity and (topological) minor exclusion. As proved in [34], model checking for FO+dp-expressible formulas can be performed in time $f(|\varphi|, |H|) \cdot |G|^2$ on H-minor-free graphs, and even more generally, as shown in [57], in time $f(|\varphi|, |H|) \cdot |G|^3$ on H-topological-minor-free graphs. Just as nowhere density is the limit of tractability for FO on subgraph-closed classes, it turns out that for FO+dp on subgraph-closed classes, the limit of tractability are classes with excluded topological minors, as it is already the case for FO+conn [49, Theorem 1.3].

In an attempt to make the logics even more expressive and useful for algorithmic purposes, another enhancement of FO was introduced by Fomin, Golovach, Sau, Stamoulis, and Thilikos in [30], the so-called *compound logic* $\tilde{\Theta}^{\mathsf{dp}}$, which combines syntactic fragments of FO+dp and CMSO and is essentially a language tailored to express general families of graph modification problems via a modification/target machinery. As proved in [30], model checking for $\tilde{\Theta}^{\mathsf{dp}}$-expressible formulas can be done in time $f(|\varphi|, |H|) \cdot |G|^2$ on H-minor-free graphs.

Fragmenting CMSO by annotated graph parameters. The aforementioned approaches are based on extensions of FO. In [55], a different perspective is proposed, which allows these and other enhanced logics to be viewed as fragments of CMSO by restricting the use of set quantification. In particular, [55] introduces *annotated treewidth logic*, denoted CMSO/atw, which is a fragment of CMSO in which second-order quantification is constrained to sets of restricted treewidth (formally, we do not measure the treewidth of the graph induced by the quantified set X, but the treewidth of X-minors to obtain a robust theory, the formal definition is given shortly). This logic is tractable on H-minor-free graph classes.

The idea of structurally restricting set quantification also appears in recent work on *low-rank* MSO, which is defined as the fragment of MSO where set quantification is limited to vertex sets of bounded cut-rank [6]. Somewhat surprisingly, over sparse graph classes, this fragment has the same expressive power as separator logic. Moreover, this logic is also applicable to dense graph classes, where it corresponds to related formalisms such as *flip-connectivity logic* and *flip-reachability logic*, which adapt the separator paradigm to settings in which vertex separators are no longer useful. Crucially, interpreting these formalisms as low-rank fragments of MSO yields a unifying semantic framework: rather than relying on ad-hoc extensions of FO, one obtains a hierarchy of MSO *fragments* whose structural restrictions simultaneously govern expressiveness and algorithmic tractability.

Finding elegant logics. From our perspective, both first-order logic and monadic second-order logic are extremely natural formalisms for reasoning about graphs and combinatorial structures. Over the years, they have proven to be robust, expressive, and conceptually clean, and a large body of algorithmic meta-theorems is built on them. Consequently, we also perceive extensions of FO, such as enrichments by connectivity or the disjoint-paths predicate, as natural logical operations. Similarly, we view restrictions of CMSO, in particular, restrictions on the sets over which one is allowed to quantify, as conceptually clean and easy to understand. In contrast, highly engineered formalisms such as compound logic are extremely powerful and very successful from an algorithmic point of view, but from a logical perspective, they appear more as programming languages than as elegant logics.

The aim of our present work is to study *elegant* restrictions of CMSO quantification that preserve both conceptual clarity and algorithmic power. We propose to organize such restrictions via *annotated graph parameters* and, in particular, via minor-monotone parameters, which, as we will demonstrate, enjoy strong compositionality properties and are well-aligned with dynamic programming techniques. We believe that the search for suitable parameters is, to a large extent, a search for elegance: different applications may suggest different parameters, and our framework is intended to support this flexibility. In this paper, we focus on two concrete parameters and show that they are functionally equivalent; one of them is particularly convenient to use, while the other is better suited for tractability arguments. More generally, our goal is to provide a uniform framework in which such parameters can be studied and compared.

Getting more technical: parametric fragments of CMSO. To present our approach and our results in full generality, let us introduce some basic notions related to annotated graphs.

An *annotated graph* is a pair (G, X), where G is a graph and X is a subset of its vertex set. An *annotated graph parameter* is a function p that maps annotated graphs to non-negative integers.

Given an annotated graph parameter p, the logic $\mathsf{CMSO/p}$ is defined by restricting existential quantification $\exists X\, \varphi$ to $\exists_{\mathsf{p}\leq k} X\, \varphi$, and universal quantification $\forall X \varphi$ to $\forall_{\mathsf{p}\leq k} X \varphi$, for any positive integer k. In such formulas, the set variable X is only allowed to be interpreted with vertex sets S such that $\mathsf{p}(G, S) \leq k$. The enhancement of $\mathsf{CMSO/p}$ by the disjoint-paths predicate is denoted by $\mathsf{CMSO/p{+}dp}$. Note that this formalism provides an extremely easy to use fragment of CMSO (CMSO/p+dp). We emphasize that the bounds appearing in parameter-bounded quantifiers are part of the formula. In particular, the size $|\varphi|$ of a formula φ depends on the largest value of k used in its quantified set variables.

A very simple example of an annotated graph parameter is the size of X, that is, $\mathsf{size}(G, X) = |X|$. It is easy to see that CMSO/size+dp has the same expressive power as FO+dp, as size-restricted set quantification can be expressed by explicit vertex quantification. The size of the resulting formula depends on the integers k that appear in the CMSO/size+dp formulas. This equivalence allows us to state the main result of [57] as follows.

▶ **Proposition 1.** *There exist a computable function $f\colon \mathbb{N}^2 \to \mathbb{N}$ and an algorithm that, given a formula $\varphi \in$ CMSO/size+dp, and an H-topological-minor-free graph G, decides whether $G \models \varphi$ in time $f(|\varphi|, |H|) \cdot |G|^3$.*

An annotated graph (H, Y) is a *minor* of (G, X), denoted $(H, Y) \leq (G, X)$, if there exists a collection $\mathcal{B} = \{B_v \mid v \in V(H)\}$ of pairwise disjoint connected subsets of $V(G)$ such that (i) for every edge $\{x, y\} \in E(H)$, the set $B_x \cup B_y$ is connected in G, and (ii) for every $y \in Y$, we have $B_y \cap X \neq \emptyset$. We refer to $\mathcal{B}$ as a *minor model* of (H, Y) in (G, X). We refer to B_v as the *branch set* of v. We call H an *X-minor* of (G, X) if $(H, V(H)) \leq (G, X)$. An annotated graph parameter is *minor-monotone* if $(H, Y) \leq (G, X)$ implies $\mathsf{p}(H, Y) \leq \mathsf{p}(G, X)$.

A much more relevant structural measure for X than its size is its treewidth, which in the context of annotated graphs leads to the notion of *annotated treewidth*: the maximum treewidth of any X-minor of G. Intuitively, annotated treewidth measures how much global structural complexity of G is witnessed by the set X. We write atw for the parameter annotated treewidth.

In graph minor theory, treewidth is characterized, up to polynomial equivalence, by the size of the largest grid minor. The $(k \times k)$-grid is the canonical model of two-dimensionality and the fundamental obstruction to bounded treewidth. Measuring the largest grid that can be obtained as a minor with all its vertices witnessed by X therefore captures, in a robust and invariant way, how "two-dimensionally" the set X sits inside G. Formally, the bidimensionality of X in G is the maximum integer k such that $(\Gamma_k, V(\Gamma_k)) \leq (G, X)$, where Γ_k is the $(k \times k)$-grid (see [61]). The *biggest rainbow grid* of (G, X), denoted by $\mathsf{brg}(G, X)$, is also called the *bidimensionality* of X in G. Because of the grid exclusion theorem [12, 51], the parameters atw and brg turn out to be functionally equivalent, that is, they functionally bound each other. It is easy to see that for every (G, X), $\mathsf{atw}(G, X) \leq \mathsf{size}(G, X)$, and, in fact, it can be arbitrarily smaller. Therefore, CMSO/atw+dp is much more expressive than CMSO/size+dp (that has the same expressive power as FO+dp).

Both atw and brg are minor-monotone annotated parameters. In Lemma 7, we show that if p is a minor-monotone annotated graph parameter, then for every fixed k, the property $\mathsf{p}(G, X) \leq k$ is definable in FO+dp using an FO+dp formula that treats the set X as a given (interpretation of) a unary relation symbol. As a consequence, the exact choice of the parameter p does not matter, as we can replace it by any functionally equivalent parameter and restrict the quantified sets appropriately by FO+dp formulas. This, applied to atw and brg, implies that CMSO/atw+dp and CMSO/brg+dp have the same expressive power. The aforementioned translation relies on the well-quasi-ordering theorem for annotated graph minors [50] and, in general, yields a non-constructive translation.

It is important to note that we do *not* say here that CMSO/p+dp collapses to FO+dp. For general instantiations of p, the logic CMSO/p+dp allows quantification over sets of *unbounded* size, subject only to a bound on the parameter p, a feature that cannot be simulated in FO+dp alone. In other words, while the property $\mathsf{p}(G, X) \leq k$ for a given set X is FO+dp-expressible, the set X itself cannot be quantified using FO+dp. We view this restricted form of set quantification as a natural and very convenient restriction of CMSO+dp, which is very well suited for practical use in expressing parameter-bounded combinatorial structures.

In this work, we will take the perspective of treewidth and in the following, restate results that were proved for atw. In particular, we can state the result of Sau, Stamoulis, and Thilikos in [55] for CMSO/ brg+dp as follows.

▶ **Proposition 2.** *There exist a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ and an algorithm that, given a formula $\varphi \in$ CMSO/atw+dp (or, equivalently, $\varphi \in$ CMSO/ brg+dp), and an H-minor-free graph G, decides whether $G \models \varphi$ in time $f(|\varphi|, |H|) \cdot |G|^2$.*

As observed in [55], the formulas of the compound logic $\tilde{\Theta}^{\mathsf{dp}}$ defined in [30] can be expressed in CMSO/atw+dp. Therefore, Proposition 2 subsumes all previous results on model checking for H-minor-free graphs. Moreover, in [55], it was also argued that the choice of brg, that is, quantifying over sets of bounded bidimensionality, is optimal in Proposition 2, in the sense that, for every choice of p, that is not parametrically "weaker than brg", CMSO/p is as expressive as CMSO on planar graphs. As CMSO can define NP-hard problems (such as 3-colorability) on planar graphs, we cannot expect tractability in this case. On the other hand, it remains an interesting question whether H-minor-free graphs constitute the combinatorial horizon of model checking for CMSO/atw+dp.

Notice that Proposition 2 and Proposition 1 are incomparable. Indeed, Proposition 2 applies to the more expressive logic CMSO/atw+dp, whereas Proposition 1 holds under the more general combinatorial assumption that a fixed graph is excluded as a topological minor. To the best of our knowledge, prior to our main result (presented below), CMSO/size+dp was the most expressive logic currently known to admit tractable model checking on graph classes excluding a fixed topological minor. Furthermore, it is known that this result cannot be extended to more general subgraph-closed graph classes [56].

Torso treewidth: an annotated parameter between annotated treewidth and size. In our present work, we study the following parameter called *torso treewidth*. Given an annotated graph (G, X), the *torso* of X in G is the graph obtained from $G[X]$ by adding, for every connected component C of $G - X$, all edges that make the neighbors of $V(C)$ in X a clique. The *torso treewidth* of (G, X), denoted by $\mathsf{ttw}(G, X)$, is defined as the minimum k such that X is contained in a vertex set whose torso has treewidth at most k. Clearly, $\mathsf{atw}(G, X) \leq \mathsf{ttw}(G, X) \leq \mathsf{size}(G, X)$, hence CMSO/ttw+dp is a fragment of CMSO/atw+dp.

This parameter for annotated graphs has already been proposed by Jansen and Swennenhuis in [42] under the name *X-free treewidth* of G, and by Hodor, La, Micek, and Rambaud in [41] under the notation $\mathsf{tw}(G, X)$. Moreover, a "multi-annotated" version of torso treewidth has been recently introduced by Protopapas, Thilikos, and Wiederrecht in [50] for colorful graphs (that is, graphs with multiple annotated sets) under the name *restricted treewidth*.

Our results. We study the annotated parameter ttw and the resulting logic CMSO/ttw+dp. We first identify an annotated parameter that relates to ttw as brg relates to atw. Given a graph G and a set $X \subseteq V(G)$, we define the *monodimensionality* of X in G as the maximum k such that $(\Gamma_k, P(\Gamma_k)) \leq (G, X)$, where $P(\Gamma_k)$ is the perimeter of Γ_k. We define the *biggest outer-annotated grid* of (G, X), denoted by $\mathsf{bog}(G, X)$, as the monodimensionality of X in G. It was proved in [41], that bog and ttw are functionally equivalent parameters.

We then show our main result, stating that we can lift the model checking result for CMSO/size+dp to CMSO/ttw+dp on classes with excluded topological minors.

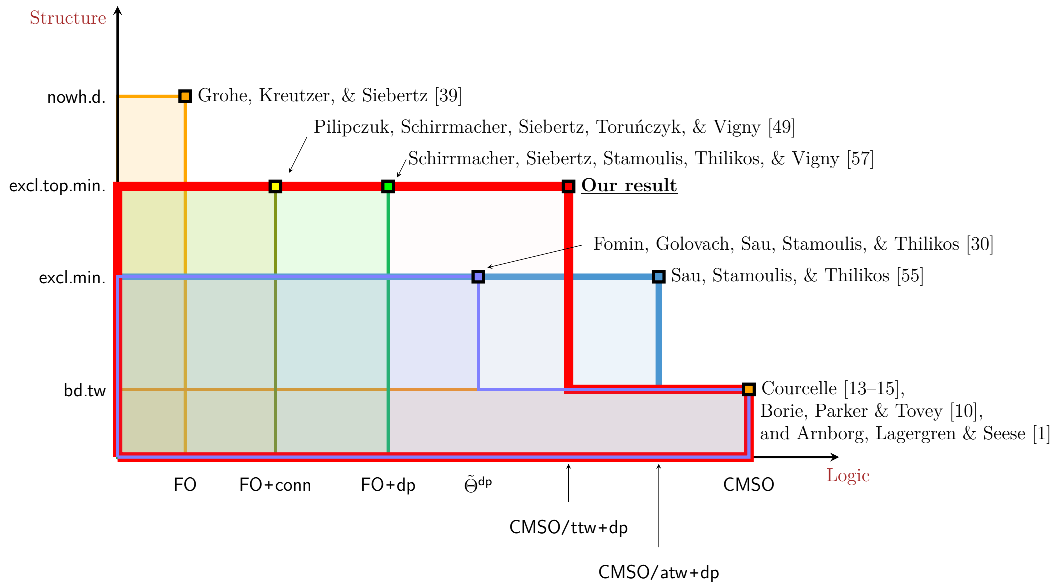

Figure 1 Our result compared to the algorithmic meta-theorems on monotone graph classes mentioned in the introduction. Throughout the paper, we focus on *monotone* graph classes, that is, classes closed under taking subgraphs. This is the natural regime for algorithmic meta-theorems based on (topological) minor exclusion. For this reason, we do not include the new tractability results for hereditary classes, such as [11, 21–24, 32]. The colorful surfaces in the figure depict the regions where the model checking for the corresponding logic is fixed-parameter tractable, and note that Courcelle's theorem [13–15] is included in a region whenever CMSO collapses to the corresponding logic on classes of graphs of bounded treewidth (for the logic introduced in this paper, cf. Section 9).

▶ **Theorem 3.** *There exist a computable function $f\colon \mathbb{N}^2 \to \mathbb{N}$ and an algorithm that, given a formula $\varphi \in$ CMSO/ttw+dp (or, equivalently, $\varphi \in$ CMSO/bog+dp), and an H-topological-minor-free graph G, decides whether $G \models \varphi$ in time $f(|\varphi|, |H|) \cdot |G|^3$.*

We also show (cf. Section 10) that Theorem 3, even for CMSO/ttw alone, is optimal in the sense that it cannot be extended to any monotone graph class that contains all graphs as topological minors. On the other hand, it is an open question whether ttw is the most "general" annotated graph parameter for which an AMT as in Theorem 3 exists.

At this point, we emphasize that torso treewidth is closer to the mechanism underlying the definition of the compound logic $\tilde{\Theta}^{\mathsf{dp}}$. Indeed, a formula of $\tilde{\Theta}^{\mathsf{dp}}$ roughly expresses the existence of a modulator X whose torso has bounded treewidth and satisfies some CMSO formula β, while $G - X$ satisfies some FO+dp formula γ. In this way, formulas in $\tilde{\Theta}^{\mathsf{dp}}$ are compound formulas in which β specifies a modification of the graph G, whereas γ specifies the target property after this modification. In [30], numerous algorithmic applications are presented for graph modification problems expressible within this modulator/target framework. It follows that every formula in $\tilde{\Theta}^{\mathsf{dp}}$ can be expressed using a bounded torso treewidth quantification over X, which readily implies that $\tilde{\Theta}^{\mathsf{dp}}$ can be viewed as a fragment of CMSO/ttw+dp. Consequently, all results of [30], together with their numerous applications to graph modification problems, also hold in the more general setting of H-topological-minor-free graph classes.

When comparing $\mathsf{CMSO/ttw{+}dp}$ and $\mathsf{FO{+}dp}$, a typical example to consider is the parameter of $\mathcal{H}$-treewidth introduced in [29]. For every minor-closed graph class $\mathcal{H}$, checking whether $\mathcal{H}$-treewidth$(G) \leq k$ can be expressed in $\mathsf{CMSO/ttw{+}dp}$ (even in $\tilde{\Theta}^{\mathsf{dp}}$) while there is evidence that this is not possible in $\mathsf{FO{+}dp}$ (see also [30] for more examples).

On the other hand, $\mathsf{CMSO/ttw{+}dp}$ can express problems that fall outside the modulator/target scheme of $\tilde{\Theta}^{\mathsf{dp}}$. For instance, one may ask whether the modulator X for which $G - X$ satisfies the target property is *unique*, as the logic also allows universal quantification over X. Moreover, universal quantification enables us to require the existence of multiple modulators X with prescribed diversity, for instance, satisfying specific Hamming-distance constraints, as suggested by the "solution diversity" framework introduced by Baste, Fellows, Jaffke, Masarík, de Oliveira Oliveira, Philip, and Rosamond in [2]. This suggests that $\mathsf{CMSO/ttw{+}dp}$ provides a conceptually simpler tool for expressing broad families of algorithmic problems. Compared to approaches that define expressibility languages via problem-definition schemes, such as the modulator/target scheme proposed in [55], it offers a more direct and uniform framework.

Our proof and the meta-theoretic value. The proof of Theorem 3 follows the general strategy developed in the model checking meta-theorems for $\mathsf{FO{+}dp}$ on topological-minor-free graph classes, but requires substantial new ingredients to handle restricted CMSO quantification. We start by computing a strongly unbreakable decomposition of the input graph using the framework of Cygan, Lokshtanov, Pilipczuk, Pilipczuk, Saurabh [18], which reduces the problem to understanding the logic on highly connected parts.

We then show that on unbreakable graphs, formulas of $\mathsf{CMSO/ttw{+}dp}$ collapse to $\mathsf{CMSO/size{+}dp}$ (which is equivalent to $\mathsf{FO{+}dp}$). More precisely, when the graph is highly connected, every set quantification over sets X in $\mathsf{CMSO/ttw{+}dp}$ is restricted to sets of bounded size, and the torso treewidth of X can be expressed in $\mathsf{FO{+}dp}$.

We then establish that $\mathsf{CMSO/ttw{+}dp}$ satisfies the *compositionality property*, which is a key property also of CMSO and FO. Informally, compositionality means that when gluing two graphs along a small interface, the type of the obtained graph depends only on the types of the two graphs that are glued. In fact, this follows from a more general result that this property is satisfied by $\mathsf{CMSO/p{+}dp}$ for *every* minor-monotone p. This is proved in Section 6 and is based on the fact that every minor-monotone annotated parameter is completely characterized by a *finite* set of obstructions (observed in [50] and based on the results in [53]). While, in general, this compositionality result is not constructive, the definitions of brg and bog (which are equivalent to atw and ttw respectively) make it constructive as both parameters are defined in terms of the exclusion of a unique parametric annotated graph, that is $(\Gamma_k, V(\Gamma_k))$ and $(\Gamma_k, P(\Gamma_k))$ respectively. The compositionality property suggests that $\mathsf{CMSO/p{+}dp}$ is a robust logical framework for expressing the algorithmic potential of fragments of CMSO and investigating what is the combinatorial horizon where they permit the proof of AMTs.

Finally, we combine these ingredients via dynamic programming over the unbreakable decomposition, using boundaried graphs and folios, as inspired by the techniques of [57], to compute and store small type-preserving representative graphs. In particular, in order to compute a small type-preserving representative, we rewrite the types in $\mathsf{FO{+}dp}$, since we know that our logic collapses to this one on unbreakable graphs, and we use the model checking algorithm of [57] to compute this representative. The unbreakability property of our decomposition does not guarantee that the graph "hanging" from each node is unbreakable, but we can guarantee unbreakability (with worse bounds) assuming that we have already computed small-size representatives for the children nodes. Let us stress here that the collapse result together with the use of the unbreakable decomposition allows for a clean reduction to the main result of [57]. Interestingly, our algorithm bypasses the explicit distinction into the "clique-minor-free" and the "large-clique-minor" cases of the algorithm of [57] by doing so implicitly when using the model checking algorithm of [57] as a blackbox. Consequently, it also avoids the direct use of the algorithm of [55] for minor-free graphs.

Overall, this yields the claimed fully constructive fixed-parameter tractable model checking algorithm for CMSO/ttw+dp on H-topological-minor-free graph classes.

From a broader perspective, our result provides a new and conceptually clean algorithmic meta-theorem principle for fragments of CMSO beyond the classical bounded-treewidth regime. It shows that structural restrictions on set quantification, formulated via minor-monotone annotated graph parameters, lead to robust and tractable logics on large graph classes, in particular, on classes excluding a fixed topological minor. This unifies and subsumes a number of previously disparate formalisms, including FO+dp and compound logic, within a single, logically natural framework. We view this as evidence that restricting CMSO quantification by elegant structural parameters is a powerful and flexible methodology for designing expressive and yet tractable logics, and that the search for suitable parameters is a fruitful direction for future research on algorithmic meta-theorems.

The use of the disjoint-paths predicate. As a final note, we would like to discuss the presence of the disjoint-paths predicate in our logic. This is motivated principally by two reasons. The first one is related to previous work: as we mentioned, part of our motivation is to generalize the meta-theorems of [30, 57] and provide a unified way of expressing all problems captured by these meta-theorems. For this, and in order to compare our logic with FO+dp, $\tilde{\Theta}^{\mathsf{dp}}$, (and even CMSO/atw+dp), it seems natural to include the disjoint-paths predicate as a feature of our logical framework. From a more technical point of view, the disjoint-paths predicate seems to play an essential role in the proof of compositionality of our logic (Lemma 19) and, therefore, in our model checking algorithm. In fact, even for the restricted fragment CMSO/ttw, our approach for model checking seems to need the stronger assumption of working with CMSO/ttw+dp to go through. Namely, for CMSO/ttw it seems essential to compute type-preserving representatives for the richer fragment CMSO/ttw+dp, because only these encode the information needed to show correctness of our replacement step. These are a couple of reasons why the disjoint-paths predicate seems to be an integral part of our approach. However, it is quite natural to ask whether its use comes with the cost of imposing hardness of model checking on monotone classes that do not exclude a topological minor. We answer in Section 10 that this is not the case: hardness also holds even in the absence of the disjoint-paths predicate.

Organization of the paper. After the introduction of needed notation for the technical results in Section 2, Section 3 and 4 are devoted to further discussions on the graph parameters and their collapse on unbreakable graphs. Then in Section 5, 6, and 7, we are providing key ingredients for our model checking algorithm, presented in Section 8. Before concluding the paper, we finally discuss the expressive power of our logic in Section 9 and the hardness of the model checking problem on classes admitting efficient encoding of topological minors in Section 10. We conclude the article in Section 11 with some directions for further research. All proofs of this extended abstract are omitted and are made available in the full version of the paper [54].

2 Preliminaries

Sets and integers. We denote by $\mathbb{N}$ the set of non-negative integers. For an integer $p \geq 1$, we set $[p] = \{1, \ldots, p\}$ and $\mathbb{N}_{\geq p} = \mathbb{N} \setminus [0, p-1]$. Also, given a function $f\colon A \to B$, we always consider its extension $f\colon 2^A \to B$ such that for every $X \subseteq A$, $f(X) = \{f(a) \mid a \in X\}$. Given two functions $\chi, \psi\colon \mathbb{N} \to \mathbb{N}$, some $r \in \mathbb{N}_{\geq 1}$, and a sequence $x_1, \ldots, x_r$ of variables, we write $\chi(n) = \mathcal{O}_{x_1,\ldots,x_r}(\psi(n))$ in order to denote that there exists a function $f\colon \mathbb{N}^r \to \mathbb{N}$ such that $\chi(n) = \mathcal{O}(f(x_1, \ldots, x_r) \cdot \psi(n))$.

2.1 Graphs and graph parameters

Graphs and annotated graphs. All graphs in this paper are finite, undirected graphs without loops and without multi-edges. We do not distinguish between isomorphic graphs and write $\mathcal{G}_{\mathsf{all}}$ for the class of all graphs. We write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. We write $\|G\|$ for $|V(G)| + |E(G)|$. For a vertex subset $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced by X.

An *annotated graph* is a pair (G, X), where $X \subseteq V(G)$. Two annotated graphs (G, X) and (G', X') are isomorphic if there is an isomorphism $\theta\colon V(G) \to V(G')$ from G to G' such that $\theta(X) = X'$. An annotated graph (H, Y) is an *annotated subgraph* of an annotated graph (G, X) if H is a subgraph of G and $Y \subseteq X$. We denote by $\mathcal{A}_{\mathsf{all}}$ the class of all annotated graphs.

Throughout the paper, when we speak about annotated graphs and formulas (from any logic), we have a signature extended by a unary predicate encoding the annotated set. We will not make the signature explicit every time but assume it tacitly.

Connectivity and disjoint paths. Let G be a graph and $u, v \in V(G)$. A u-v-path P in G is a sequence $v_1, \ldots, v_\ell$ of pairwise different vertices such that $\{v_i, v_{i+1}\} \in E(G)$ for all $1 \leq i < \ell$ and $v_1 = u$ and $v_\ell = v$. The vertices $v_2, \ldots, v_{\ell-1}$ are the *internal vertices* of P and the vertices u and v are its *endpoints*. Two vertices u, v are *connected* if there exists a path with endpoints u, v. A graph is *connected* if every two of its vertices are connected and a set $X \subseteq V(G)$ is *connected* in G if $G[X]$ is connected. Two paths P, Q are *internally vertex-disjoint* if no vertex of one path appears as an internal vertex of the other path. They are *disjoint* if their vertex sets are disjoint.

Let G be a graph, $r \in \mathbb{N}$, and $s_1, t_1, \ldots, s_r, t_r$ pairwise distinct vertices of G. Then $\mathsf{dp}_r(s_1, t_1, \ldots, s_r, t_r)$ is true in G if and only if there are disjoint paths between s_i and t_i for $1 \leq i \leq r$. We use dp instead of dp_r when r is clear from the context.

Minors and topological minors. Given a graph G and some $S \subseteq V(G)$, we say that S is a *connected* set if $G[S]$ is a connected graph. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A *minor model* of H in G is collection $\mathcal{B} = \{B_v \mid v \in V(H)\}$ of pairwise-disjoint connected subsets of $V(G)$ where, for every $\{u, v\} \in E(H)$, $B_u \cup B_v$ is connected. Each set B_v is called the *branch set* of v.

Given two annotated graphs (H, Y) and (G, X), we say that (H, Y) is a *minor* of (G, X), denoted by $(H, Y) \leq (G, X)$, if there exists a minor model $\mathcal{B}$ of H in G such that for every $u \in Y$, we have $B_u \cap X \neq \emptyset$.

The *dissolution* of a vertex v of degree two whose two neighbors are x and y is the operation of removing v and adding the edge $\{x, y\}$ (if it does not already exist). A graph H is a *topological minor* of a graph G if H can be obtained from a subgraph of G by dissolving edges.

A *topological minor model* of a graph H in a graph G is an injective mapping η that maps vertices of H to vertices of G and edges of H to pairwise internally vertex-disjoint paths in G so that for every $\{u,v\} \in E(H)$ the path $\eta(\{u,v\})$ has the endpoints $\eta(u)$ and $\eta(v)$. The vertices in $\{\eta(v) \mid v \in V(H)\}$ are called *principal vertices* of the model in G. A graph H is a *topological minor* of G if there is a topological minor model of H in G. We say that an annotated graph (H,Y) is a *topological minor* of an annotated graph (G,X), denoted by $(H,Y) \preceq (G,X)$, if there exists a topological minor model η of H in G such that $\eta(Y) \subseteq X$.

We call a graph G *H-minor-free*, and *H-topological-minor-free*, respectively, if H is not a minor, or topological minor of G. We call a class $\mathscr{C}$ of graphs *(topological-)minor-free* if there exists a graph H such that every member of $\mathscr{C}$ is H-(topological-)minor-free.

Given an annotated graph (H,Y), we define $\mathsf{ext}(H,Y)$ as the set of all topological-minor minimal annotated graphs (G,X) that contain (H,Y) as an annotated minor. We need the following easy observation.

▶ **Observation 4.** *Let (H,Y) be an n-vertex annotated graph. Then, every graph in $\mathsf{ext}(H,Y)$ has at most $\mathcal{O}(n^2)$ vertices.*

When any definition for annotated (topological) minors can be applied in the case of empty annotation, i.e., $X = \emptyset$, then, for simplicity, we drop the term "annotated".

Annotated graph parameters. An *annotated graph parameter* is a function $\mathsf{p}\colon \mathcal{A}_{\mathsf{all}} \to \mathbb{N}$, mapping annotated graphs to non-negative integers.

▶ **Observation 5.** *Let $\mathsf{p}\colon \mathcal{A}_{\mathsf{all}} \to \mathbb{N}$ be a minor-monotone annotated graph parameter and let (G,X) be an annotated graph. Then, for all $Y \subseteq X$, it holds that $\mathsf{p}(G,Y) \leq \mathsf{p}(G,X)$.*

Obstructions. Given a minor-monotone annotated graph parameter $\mathsf{p}\colon \mathcal{A}_{\mathsf{all}} \to \mathbb{N}$, we define $\mathcal{O}_k^{\mathsf{p}}$ as the set of all $\leq$-minimal annotated graphs where the value of p exceeds k. The following lemma is based on [50, Theorem 1.1] that, in turn, is based on the results of Robertson and Seymour in [53].

▶ **Lemma 6.** *For every minor-monotone annotated graph parameter $\mathsf{p}\colon \mathcal{A}_{\mathsf{all}} \to \mathbb{N}$, there is a function $f_{\mathsf{p}}\colon \mathbb{N} \to \mathbb{N}$ and a collection of finite sets $\{\mathcal{O}_k^{\mathsf{p}} \mid k \in \mathbb{N}\}$ such that, for every $k \in \mathbb{N}$, $|\mathcal{O}_k^{\mathsf{p}}| \leq f_{\mathsf{p}}(k)$ and for every annotated graph (G,X), $\mathsf{p}(G,X) \leq k$ if and only if (G,X) excludes all graphs in $\mathcal{O}_k^{\mathsf{p}}$ as topological minors.*

Based on Lemma 6, we show that every minor-monotone annotated graph parameter can be expressed in $\mathsf{FO{+}dp}$.

▶ **Lemma 7.** *For every minor-monotone annotated graph parameter $\mathsf{p}\colon \mathcal{A}_{\mathsf{all}} \to \mathbb{N}$ and every $k \in \mathbb{N}$, there is a formula $\sigma_k^{\mathsf{p}} \in \mathsf{FO{+}dp}$ such that $(G,X) \models \sigma_k^{\mathsf{p}}$ iff $\mathsf{p}(G,X) \leq k$.*

Equivalent (annotated) parameters. Let p and p' be two annotated graph parameters. We say that $\mathsf{p} \preceq \mathsf{p}'$ if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every annotated graph (G,X), it holds that $\mathsf{p}(G,X) \leq f(\mathsf{p}'(G,X))$. We say that p and p' are *functionally equivalent*, or simply *equivalent*, which we denote by $\mathsf{p} \sim \mathsf{p}'$, if $\mathsf{p} \preceq \mathsf{p}'$ and $\mathsf{p}' \preceq \mathsf{p}$. The parameter ordering relation $\preceq$ and the concept of annotated parameter equivalence extend to non-annotated parameters in the obvious way.

2.2 Tree decompositions and unbreakability

Trees and tree orders. A *tree* is an acyclic and connected graph T. By assigning a distinguished vertex r as the root of a tree, we impose a tree order $\preceq_T$ on $V(T)$ by $x \preceq_T y$ if x lies on the unique path (possibly of length 0) from y to r. If $x \preceq_T y$, we call x an *ancestor* of y in T. Note that by this definition, every node is an ancestor of itself. We drop the subscript T if it is clear from the context. We write $\mathsf{parent}(x)$ for the parent of a non-root node x of T, and $\mathsf{children}(x)$ for the set of children of x in T. We define $\mathsf{parent}(r) = \perp$.

Tree decompositions. A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, \mathsf{bag})$, where T is a rooted tree and $\mathsf{bag}\colon V(T) \to 2^{V(G)}$ is a mapping assigning to each node x of T its *bag* $\mathsf{bag}(x)$, which is a subset of vertices of G such that the following conditions are satisfied:

1. For every vertex $u \in V(G)$, the set of nodes $x \in V(T)$ with $u \in \mathsf{bag}(x)$ induces a nonempty and connected subtree of T.
2. For every edge $\{u, v\} \in E(G)$, there exists a node $x \in V(T)$ with $\{u, v\} \subseteq \mathsf{bag}(x)$.

The *width* of a tree decomposition is the value $\max_{t \in V(T)} |\mathsf{bag}(t)| - 1$ and the *treewidth* of a graph G is the minimum possible width of a tree decomposition of G.

Recall that if r is the root of T, then we have $\mathsf{parent}(r) = \perp$. We define $\mathsf{bag}(\perp) = \emptyset$. For a node $x \in V(T)$, we define the *adhesion* of x as $\mathsf{adh}(x) \coloneqq \mathsf{bag}(\mathsf{parent}(x)) \cap \mathsf{bag}(x)$; the *margin* of x as $\mathsf{mrg}(x) \coloneqq \mathsf{bag}(x) \setminus \mathsf{adh}(x)$; the *cone at* x as $\mathsf{cone}(x) \coloneqq \bigcup_{y \succeq_T x} \mathsf{bag}(y)$; and the *component at* x as $\mathsf{comp}(x) \coloneqq \mathsf{cone}(x) \setminus \mathsf{adh}(x)$. We will see in a moment that we may assume that the component at x is connected for every node x, which justifies the name component.

The *adhesion* of a tree decomposition $\mathcal{T} = (T, \mathsf{bag})$ is defined as the largest size of an adhesion, that is, $\max_{x \in V(T)} |\mathsf{adh}(x)|$. Given a tree decomposition $\mathcal{T} = (T, \mathsf{bag})$, we refer to T as the *underlying tree* of $\mathcal{T}$.

A tree decomposition $\mathcal{T} = (T, \mathsf{bag})$ is *regular* if for every non-root node $x \in V(T)$

1. the margin $\mathsf{mrg}(x)$ is nonempty;
2. the graph $G[\mathsf{comp}(x)]$ is connected; and
3. every vertex of $\mathsf{adh}(x)$ has a neighbor in $\mathsf{comp}(x)$.

Unbreakability. A *separation* in a graph G is a pair (A, B) of subsets of vertices of G such that $A \cup B = V(G)$ and there is no edge with one endpoint in $A \setminus B$ and the other endpoint in $B \setminus A$. The *separator* of G associated with (A, B) is the intersection $A \cap B$ and the *order* of a separation is the size of its separator.

For $q, k \in \mathbb{N}$, a vertex subset X in a graph G is (q, k)*-unbreakable* if for every separation (A, B) of G of order at most k, we have

$$|A \cap X| \le q \qquad \text{or} \qquad |B \cap X| \le q.$$

We say that a graph G is (q, k)*-unbreakable* if $V(G)$ is (q, k)-unbreakable in G.

The notion of unbreakability can be lifted to tree decompositions by requiring it from every individual bag. In [18], Cygan, Lokshtanov, Pilipczuk, Pilipczuk, and Saurabh introduce two notions of unbreakable tree decompositions, which differ on whether we impose each bag to be unbreakable in its cone, or in the entire graph. These definitions then yield different bounds and computation time. We use the notion named *strongly unbreakable*.

For $q, k \in \mathbb{N}$, a tree decomposition $\mathcal{T} = (T, \mathsf{bag})$ of a graph G is *strongly* (q, k)*-unbreakable* if for every $x \in V(T)$, $\mathsf{bag}(x)$ is (q, k)-unbreakable in $G[\mathsf{cone}(x)]$.

▶ **Proposition 8** ([18]). *There is a function $q(k) \in 2^{\mathcal{O}(k)}$ such that for every graph G and $k \in \mathbb{N}$, there exists a strongly $(q(k), k)$-unbreakable tree decomposition of G of adhesion at most $q(k)$. Moreover, given G and k, such a tree decomposition can be computed in time $2^{\mathcal{O}(k^2)} \cdot |G|^2 \cdot \|G\|$.*

Given any strongly (q, k)-unbreakable decomposition, we can refine it so that it becomes regular (and remains strongly unbreakable), see [5]. Hence, we may assume that the tree decompositions constructed by the algorithm of Proposition 8 are regular.

2.3 Fragments of counting monadic second-order logic

Counting monadic second-order logic. We use *counting monadic second-order logic* (CMSO) with quantification over vertex sets, which extends monadic second-order logic (MSO) by modulo counting predicates on set cardinalities. For conceptual clarity, we work over a relational signature σ consisting of a single binary relation symbol E (the edge relation) and a finite set of unary relation symbols, called *colors*. A *colored graph* is a σ-structure such that E is interpreted as a symmetric and irreflexive relation on the universe, and each unary relation symbol is interpreted as a subset of elements of the universe. Quantification over edge sets (CMSO_2) or tuples of relations (as in guarded CMSO) can be simulated for general relational signatures, by considering the incidence encoding. By considering the incidence encoding of structures, we can also apply the disjoint-paths predicate in the more general setting.

In the context of graphs, MSO and its counting extension CMSO were systematically studied in the work of Courcelle and Engelfriet; a standard reference is [16]. The original application of MSO to graphs appears in Courcelle's classic paper [13]. We assume basic knowledge of logic and refer the reader to the textbook [47] for more background.

CMSO is defined as follows. The variables of the logic are first-order variables $x, y, z, \ldots$ ranging over vertices, and set variables $X, Y, Z, \ldots$ ranging over subsets of vertices. The formulas of CMSO are built from the atomic formulas $x = y$, $E(x, y)$, $C(x)$ for each unary relation symbol $C \in \sigma$, $x \in X$, $\mathsf{card}(X) \equiv i \pmod m$ for fixed integers $m \geq 1$ and $0 \leq i < m$, using the Boolean connectives $\neg, \wedge, \vee$ and the quantifiers $\exists x, \forall x, \exists X, \forall X$. We use $\forall X\, \varphi$ as an abbreviation for $\neg\exists X\, \neg\varphi$.

The semantics are the standard Tarskian semantics for relational structures. In particular, set variables range over all subsets of the universe, and the counting atom $\mathsf{card}(X) \equiv i \pmod m$ is true if and only if the cardinality of the interpretation of X is congruent to i modulo m.

Syntax and semantics of CMSO/p. Let $\mathsf{p} \colon \mathcal{A}_{\mathsf{all}} \to \mathbb{N}$ be an annotated graph parameter. The terms, (atomic) formulas, and free variables of CMSO/p are defined as the ones of CMSO with the only exception that instead of using the quantifier $\exists X$, where X is a set variable, we use the quantifier $\exists_{\mathsf{p}\leq k} X$ (semantically defined below) for some $k \in \mathbb{N}$. Also, we use $\forall_{\mathsf{p}\leq k} X\varphi$ as an abbreviation of $\neg(\exists_{\mathsf{p}\leq k} X \neg\varphi)$. The semantics of CMSO/p are as the ones of CMSO, where a colored graph G satisfies $\exists_{\mathsf{p}\leq k} X\varphi$, denoted by $G \models \exists_{\mathsf{p}\leq k} X\varphi$, if and only if there is a vertex subset $S \subseteq V(G)$ such that $\mathsf{p}(G, S) \leq k$ and G satisfies φ when X is interpreted as S. We define $|\varphi|$ as the encoding length of φ. In particular, this takes into account the values k appearing in the quantifiers of φ as well as the values i and m appearing in atomic formulas of the form $\mathsf{card}(X) \equiv i \pmod m$ in φ.

The logic CMSO/p+dp. For every $k \in \mathbb{N}$, we define the $2k$-ary *disjoint-paths predicate* $\mathsf{dp}(x_1, y_1, \ldots, x_k, y_k)$, where $x_1, y_1, \ldots, x_k, y_k$ are first-order terms. Although the following definition makes sense for any logic $\mathcal{L}$, we will only consider it for extensions of $\mathsf{CMSO/p}$, which is the focus of this paper.

A formula of $\mathsf{CMSO/p+dp}$ is obtained by enhancing the syntax of $\mathsf{CMSO/p}$ by allowing atomic formulas of the form $\mathsf{dp}(x_1, y_1, \ldots, x_k, y_k)$ on first-order terms $x_1, y_1, \ldots, x_k, y_k$, for $k \in \mathbb{N}$. The satisfaction relation between a colored graph G and formulas φ of $\mathsf{CMSO/p+dp}$ is as for $\mathsf{CMSO/p}$, where $\mathsf{dp}(x_1, y_1, \ldots, x_k, y_k)$ evaluates true in G if and only if there are pairwise vertex disjoint paths $P_1, \ldots, P_k$ in G between (the interpretations of) x_i and y_i for all $i \in [k]$.

It is not difficult to see that $\mathsf{dp}(x_1, y_1, \ldots, x_k, y_k)$ is CMSO-expressible for every fixed $k \in \mathbb{N}$. Suppose now that p is minor-monotone. Then because of Lemma 7, the query $\mathsf{p}(G, X) \leq k$ can be expressed in $\mathsf{FO+dp}$, and therefore also in CMSO. This implies that for every minor-monotone p, the logic $\mathsf{CMSO/p+dp}$ is contained in CMSO.

Using Lemma 7, we prove the following.

▶ **Lemma 9.** *If* p *and* p' *are equivalent annotated graph parameters that are both minor-monotone, then the logics* $\mathsf{CMSO/p+dp}$ *and* $\mathsf{CMSO/p'+dp}$ *have the same expressive power.*

Prenex normal form. In the rest of this paper, we assume that for every considered formula φ, the maximum m such that a predicate of the form $\mathsf{card}(X) \equiv i \pmod{m}$ appears in φ is bounded by a fixed constant.

We say that a sentence φ of $\mathsf{CMSO/p+dp}$ is in *prenex normal form* if it is of the form

$$Q_1 X_1 \ \ldots Q_\chi X_\chi \ Q_{\chi+1} x_{\chi+1} \ \ldots Q_{\chi+\beta} x_{\chi+\beta} \ \psi(X_1, \ldots, X_\chi, x_{\chi+1}, \ldots, x_{\chi+\beta}),$$

where $\chi, \beta \in \mathbb{N}$ and

- $X_1, \ldots, X_\chi$ are vertex set variables and $x_{\chi+1}, \ldots, x_{\chi+\beta}$ are vertex variables;
- for each $i \in [\chi]$, $Q_i \in \{\exists_{\mathsf{p} \leq w}, \forall_{\mathsf{p} \leq w}\}$, for some $w \in \mathbb{N}$;
- for each $i \in [\chi+1, \chi+\beta]$, $Q_i \in \{\exists, \forall\}$; and
- $\psi(X_1, \ldots, X_\chi, x_{\chi+1}, \ldots, x_{\chi+\beta})$ is a quantifier-free formula of $\mathsf{CMSO/p+dp}$.

For a given $i \in [\chi+\beta]$, the *ith variable of* φ is X_i if $i \leq \chi$ and x_i if $i \geq \chi+1$. We refer to $\chi+\beta$ as the *quantifier rank* of φ. We assume that every given sentence of $\mathsf{CMSO/bog+dp}$ is in prenex normal form. Let φ be a formula of $\mathsf{CMSO/p+dp}$. The dp-*rank* of φ is the maximum integer r such that the predicate dp_r appears in φ, or 0 if there is no dp predicate in φ. Also, the p-*rank* of φ is the maximum integer w such that $\exists_{\mathsf{p} \leq w}$ or $\forall_{\mathsf{p} \leq w}$ appears in φ, or 0 if no such quantifier is used in φ.

3 Torso treewidth and monodimensionality

In this section, we introduce the annotated parameter *torso treewidth* and we present its parametric equivalence with the concept of monodimensionality. In the full version of the paper we give another equivalent parameter, based on bidimensionality and the "attachment degree" of a set.

Torso treewidth. Recall that, given an annotated graph (G, X), the *torso* of X in G is the graph obtained from $G[X]$ by adding, for every connected component C of $G - X$, all edges that make the neighbors of $V(C)$ in X a clique. We define the *torso treewidth* of an annotated graph (G, X) as

$$\mathsf{ttw}(G, X) = \min\{\mathsf{tw}(\mathsf{torso}(G, X')) \mid X \subseteq X'\}.$$

The minimization over all supersets $X' \supseteq X$ in the definition of ttw is crucial. Intuitively, X is intended to represent the "interface" to the remainder of the graph, and we are allowed to enlarge it by adding additional vertices so as to obtain a torso of small treewidth. Formally, this minimization is needed to ensure that ttw is robust and, in particular, minor-monotone. As a simple example, let $G = K_{1,k}$ and let X be the set of leaves. Then, $\mathsf{torso}(G, X)$ is a clique on k vertices, hence $\mathsf{tw}(\mathsf{torso}(G, X)) = k - 1$. However, the intended "complexity witnessed by X" should not depend on whether we include the unique high-degree vertex c in the interface. Setting $X' := X \cup \{c\}$, we obtain $\mathsf{torso}(G, X') = K_{1,k}$ and thus, $\mathsf{tw}(\mathsf{torso}(G, X')) = 1$. Our definition captures this robustness by allowing us to add vertices to the interface when this simplifies the structure of the torso, and it is this flexibility that makes ttw behave well under taking minors. It is also easy to observe that ttw is a minor-monotone annotated graph parameter (see the full version of the paper).

The definition of torso treewidth can alternatively be derived by the concept of X-free decompositions of G recently proposed by Jansen and Swennenhuis in [42]. Also, the same annotated graph parameter has been introduced and examined by Hodor, La, Micek, and Rambaud in [41].

Monodimensionality. We introduce a final annotated graph parameter, which we already mentioned in the introduction. The *outer-annotated* $(k \times k)$*-grid* is $\mathbf{\Gamma}_k^\circ = (\Gamma_k, P_k)$, where P_k are the vertices of the perimeter (that is, the outer boundary) of Γ_k. We define the *monodimensionality* of a set X in a graph G as the maximum k such that $\mathbf{\Gamma}_k^\circ \leq (G, X)$. We write $\mathsf{bog}(G, X)$ for the monodimensionality of X in G (bog stands for biggest outer-annotated grid). By definitions bog is minor-monotone.

Using the main result of Marx, Seymour, and Wollan in [48], Hodor, La, Micek, and Rambaud [41] proved, using the terminology introduced in Section 2, that $\mathsf{ttw} \sim \mathsf{bog}$.

4 Collapse on unbreakable graphs

Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two logics and let $\mathscr{C}$ be a class of graphs. We say that $\mathcal{L}_1$ *collapses to* $\mathcal{L}_2$ *on* $\mathscr{C}$ if for every formula of $\mathcal{L}_1$, there is a formula of $\mathcal{L}_2$ such that the two formulas are equivalent on all graphs from $\mathscr{C}$.

In this section, we prove the collapse of $\mathsf{CMSO/ttw{+}dp}$ and $\mathsf{CMSO/ttw}$ to $\mathsf{FO{+}dp}$ and FO respectively, on unbreakable graphs. The first ingredient is the following lemma, which states that on unbreakable graphs, only small sets can have bounded torso treewidth.

▶ **Lemma 10.** *Let $k, q \in \mathbb{N}$ and let G be a $(q, k{+}1)$-unbreakable graph with at least $3q$ vertices. Also, let X be a vertex subset of $V(G)$ with $\mathsf{ttw}(G, X) \leq k$. Then, for every $X' \supseteq X$ with $\mathsf{ttw}(G, X) = \mathsf{tw}(\mathsf{torso}(G, X'))$, it holds that $|X'| \leq q$.*

As proved in Lemma 7, the fact that an annotated graph has small torso treewidth can be expressed in $\mathsf{FO{+}dp}$. Instead of the annotated graph (G, X), we talk here about the graph G together with free variables $x_1, \dots, x_\ell$, which is only as powerful when the set X is known to have small size, see Lemma 10.

▶ **Lemma 11.** *For all integers k, ℓ, there is an $\mathsf{FO{+}dp}$ formula $\sigma_k^{\mathsf{FO+dp}}(x_1, \dots, x_\ell)$ such that for every graph G and set $A = \{a_1, \dots, a_\ell\} \subseteq V(G)$, we have that $(G, \bar{a}) \models \psi_k$ if and only if $\mathsf{ttw}(G, A) \leq k$.*

Using the previous two lemmas, we can now show that on unbreakable graphs (with well-chosen parameters), $\mathsf{CMSO/ttw{+}dp}$ collapses to $\mathsf{FO{+}dp}$.

▶ **Lemma 12.** *There is a function $f_{\mathsf{rank}}\colon \mathbb{N}^3 \to \mathbb{N}$ such that the following holds. For all $d, k, q \in \mathbb{N}$ and φ a* CMSO/ttw+dp*-sentence in prenex normal form that has quantifier rank d and* ttw*-rank k, there is an* FO+dp*-sentence ψ of quantifier rank at most $f_{\mathsf{rank}}(d, k, q)$, such that for every $(q, k+1)$-unbreakable graph G, it holds that $G \models \varphi$ if and only if $G \models \psi$.*

It is more involved, but we can actually prove a similar collapse from CMSO/ttw to FO on unbreakable graphs. The proof of Lemma 12 is the exact same, with the only exception that $\sigma_k^{\mathsf{FO+dp}}$ now needs to be written in FO. Note that although FO+dp does not collapse to FO on unbreakable graphs, this can be achieved for these specific formulas.

▶ **Lemma 13.** *For all integers q, k, ℓ, there is a* FO *formula $\sigma_k^{\mathsf{FO}}(x_1, \ldots, x_\ell)$ such that for every (q, k)-unbreakable graph G, and set $A = \{a_1, \ldots, a_\ell\} \subseteq V(G)$, we have that $(G, \bar{a}) \models \sigma_k^{\mathsf{FO}}$ iff $\mathsf{ttw}(G, A) \leq k$.*

Thanks to Lemma 13, we can now adapt the proof of Lemma 12 to yield the following result.

▶ **Lemma 14.** *There is a function $f_{\mathsf{rank}}\colon \mathbb{N}^3 \to \mathbb{N}$ such that the following holds. For all $d, k, q \in \mathbb{N}$ and φ a* CMSO/ttw*-sentence in prenex normal form that has quantifier rank d and* ttw*-rank k, there is an* FO*-sentence ψ of quantifier rank at most $f_{\mathsf{rank}}(d, k, q)$, such that for every $(q, k+1)$-unbreakable graph G, it holds that $G \models \varphi$ if and only if $G \models \psi$.*

5 Preliminaries on folios

In this section, we define boundaried (annotated) graphs, which are (annotated) graphs with a distinguished set of vertices and a linear order among these vertices. We also define when two annotated boundaried graphs are *compatible* and the notion of *folio* of an annotated boundaried graph. Finally, we prove Lemma 17, which is the combinatorial engine of the proof of the compositionality of CMSO/p+dp in Section 6.

5.1 Boundaried annotated graphs

An important combinatorial ingredient of our proofs is boundaried graphs. This subsection is dedicated to their introduction and to the extension of the containment relations that we gave in previous sections to the boundaried setting.

Boundaried graphs. Let $t \in \mathbb{N}$. An *annotated t-boundaried graph* is a quadruple $\mathbf{G} = (G, X, B, \rho)$ where (G, X) is an annotated graph, $B \subseteq V(G)$, $|B| = t$, and $\rho\colon B \to [t]$ is a bijective function (see [20, Definition 12.9.1]). We call B the *boundary* of $\mathbf{G}$ and the vertices of B *the boundary vertices* of $\mathbf{G}$. We also call (G, X) *the underlying annotated graph* of $\mathbf{G}$ and we call G *the underlying graph* of $\mathbf{G}$. The quadruple (G, X, B, ρ) is an *annotated boundaried graph* if it is an annotated t-boundaried graph, for some $t \in \mathbb{N}$.

We say that an annotated t-boundaried graph $\mathbf{G}' = (G', X', B', \rho')$ is an (annotated boundaried) *subgraph* of $\mathbf{G}$ if G' is a subgraph of G, $B' = B$, $B \cap X = B' \cap X'$, $X' \subseteq X \cap V(G')$, and $\rho' = \rho$. Two t-boundaried graphs $\mathbf{G}_1 = (G_1, X_1, B_1, \rho_1)$ and $\mathbf{G}_2 = (G_2, X_2, B_2, \rho_2)$ are *isomorphic* if G_1 is isomorphic to G_2 via a bijection $\varphi\colon V(G_1) \to V(G_2)$ such that $\rho_1 = \rho_2 \circ \varphi|_{B_1}$, i.e., the vertices of B_1 are mapped via φ to equally indexed vertices of B_2, and, moreover, $\rho_2|_{X_2} = \rho_2 \circ \varphi|_{X_1}$, i.e., boundary vertices in X are equally indexed in both $\mathbf{G}_1$ and $\mathbf{G}_2$. As in [52] (see also [3]), we define the *detail* of an annotated boundaried graph $\mathbf{G} = (G, X, B, \rho)$ as $\mathsf{detail}(\mathbf{G}) := \max\{|E(G)|, |V(G) \setminus B|\}$. We denote by $\mathcal{A}^{(t)}$ the set of all (pairwise non-isomorphic) annotated t-boundaried graphs. We also set $\mathcal{A} = \bigcup_{t \in \mathbb{N}} \mathcal{A}^{(t)}$.

Compatible annotated boundaried graphs. Consider two annotated boundaried graphs $\mathbf{G}_1 = (G_1, X_1, B_1, \rho_1)$ and $\mathbf{G}_2 = (G_2, X_2, B_2, \rho_2)$. We say that $\mathbf{G}_1$ and $\mathbf{G}_2$ are *compatible* if $\rho_2^{-1} \circ \rho_1$ is an isomorphism from $G_1[B_1]$ to $G_2[B_2]$ such that $\rho_1(X_1) = \rho_2(X_2)$. Given two compatible annotated boundaried graphs $\mathbf{G}_1 = (G_1, X_1, B_1, \rho_1)$ and $\mathbf{G}_2 = (G_2, X_2, B_2, \rho_2)$, we define $\mathbf{G}_1 \oplus \mathbf{G}_2$ as the annotated graph obtained if we take the disjoint union of G_1 and G_2 and, for every $i \in [|B_1|]$, we identify the vertices $\rho_1^{-1}(i)$ and $\rho_2^{-1}(i)$, agreeing that after each such identification the vertices of the boundary of $\mathbf{G}_1$ prevail.

In case $X = \emptyset$, instead of $\mathbf{G} = (G, X, B, \rho)$, for simplicity, we write $\mathbf{G} = (G, B, \rho)$, and we call $\mathbf{G}$ a *boundaried graph*, and all the aforementioned definitions naturally apply to boundaried graphs as well. We also use $\mathcal{B}^{(t)}$ for the set of all (pairwise non-isomorphic) t-boundaried graphs and we set $\mathcal{B} = \bigcup_{t\in\mathbb{N}} \mathcal{B}^{(t)}$.

Topological minors of annotated boundaried graphs. If $\mathbf{M} = (M, X, B, \rho) \in \mathcal{A}$ and $T \subseteq V(M)$ with $B \subseteq T$, and such that every vertex in $V(M) \setminus T$ has degree 2, we call $(\mathbf{M}, T)$ an abtm-*pair* and we define

$$\mathsf{diss}(\mathbf{M}, T) = (\mathsf{diss}(M, T), X \cap T, B, \rho),$$

where $\mathsf{diss}(M, T)$ is the result of the dissolution in M of all vertices in $V(M) \setminus T$. We refer to the set T as the set of *branch vertices* of $(\mathbf{M}, T)$. Note that we do not permit dissolution of boundary vertices, as we demand all of them to be branch vertices. If $\mathbf{G} = (G, X, B, \rho)$ is an annotated boundaried graph, we say that $(\mathbf{M}, T)$ is an abtm-pair *of* $\mathbf{G}$ if it is an abtm-pair where $\mathbf{M}$ is a subgraph of $\mathbf{G}$ and $B \subseteq T$. Let $\mathbf{G}_1 = (G_1, X_1, B_1, \rho_1)$ and $\mathbf{G}_2 = (G_2, X_2, B_2, \rho_2)$ be two annotated boundaried graphs. We say that $\mathbf{G}_1$ is an *annotated topological minor* of $\mathbf{G}_2$, denoted by $\mathbf{G}_1 \preceq \mathbf{G}_2$, if $\mathbf{G}_1$ and $\mathbf{G}_2$ are compatible and $\mathbf{G}_2$ has an abtm-pair $(\mathbf{M}, T)$ such that $\mathsf{diss}(\mathbf{M}, T)$ is isomorphic to $\mathbf{G}_1$.

5.2 Folios

We next present the definition of folios and extended folios of (annotated) boundaried graphs, which encode the set of (annotated) topological minors they contain. These notions will be used several times in the remainder of the paper. At the end of this subsection, we prove Lemma 17, which (informally) states that the extended folio captures the partial behavior of any fixed minor-monotone annotated graph parameter on the set X of an annotated boundaried graph $\mathbf{G} = (G, X, B, \rho)$.

Folios and extended folios. Given $\mathbf{G} \in \mathcal{A}$ and $\ell \in \mathbb{N}$, we define the ℓ-*folio* of $\mathbf{G}$ as

$$\ell\text{-}\mathsf{folio}(\mathbf{G}) := \left\{ \mathbf{G}' \in \mathcal{A} \;\middle|\; \mathbf{G}' \preceq \mathbf{G} \text{ and } \mathsf{detail}(\mathbf{G}') \leq \ell \right\}.$$

The number of distinct ℓ-folios of annotated t-boundaried graphs is bounded in the following result, proved first in [4] and used also in [3].

▶ **Proposition 15** ([4]). *There exists a function $f_{15} \colon \mathbb{N}^2 \to \mathbb{N}$ such that for every $t, \ell \in \mathbb{N}$, it holds that, for every $\mathbf{G} \in \mathcal{A}^{(t)}$,*

$$|\{\ell\text{-}\mathsf{folio}(\mathbf{G}) \mid \mathbf{G} \in \mathcal{A}^{(t)}\}| \leq f_{15}(t, \ell).$$

Let $\mathbf{G} = (G, X, B, \rho)$ be an annotated boundaried graph. Given $I \subseteq \binom{[|B|]}{2}$, we write $\mathbf{G}^I$ for the (annotated) boundaried graph obtained from $\mathbf{G}$ by adding in its underlying graph the edges in

$$\left\{ \{\rho^{-1}(i), \rho^{-1}(j)\} \;\middle|\; \{i, j\} \in I \right\}.$$

We define the *extended ℓ-folio of* $\mathbf{G} = (G, X, B, \rho)$ to be

$$\ell\text{-ext-folio}(\mathbf{G}) \quad = \quad \{(I, \ell\text{-folio}(\mathbf{G}^I)) \mid I \in \binom{[|B|]}{2}\}. \tag{1}$$

From Proposition 15, and the definition in (1), we easily obtain the following.

▶ **Observation 16.** *There exists a function $f_{16} \colon \mathbb{N}^2 \to \mathbb{N}$ such that for every $t, \ell \in \mathbb{N}$, and every $\mathbf{G} \in \mathcal{B}^{(t)}$, it holds that*

$$|\ell\text{-ext-folio}(\mathbf{G})| \leq f_{16}(t, \ell).$$

We say that two (annotated) boundaried graphs $\mathbf{G}_1, \mathbf{G}_2$ are *ℓ-equivalent* if they are compatible and

$$\ell\text{-ext-folio}(\mathbf{G}_1) = \ell\text{-ext-folio}(\mathbf{G}_2).$$

After presenting the above definitions, we are now ready to state and prove the following result, whose proof uses Lemma 6.

▶ **Lemma 17.** *For every minor-monotone annotated graph parameter $\mathsf{p} \colon \mathcal{A}_{\mathsf{all}} \to \mathbb{N}$, there exists a function $f_{17}^{\mathsf{p}} \colon \mathbb{N} \to \mathbb{N}$ such that the following holds: Let $w \in \mathbb{N}$. If two annotated boundaried graphs $\mathbf{G}_1$, $\mathbf{G}_2$ are $f_{17}^{\mathsf{p}}(w)$-equivalent, then for every compatible annotated boundaried graph $\mathbf{C}$, it holds that*

$$\mathsf{p}(\mathbf{C} \oplus \mathbf{G}_1) \leq w \iff \mathsf{p}(\mathbf{C} \oplus \mathbf{G}_2) \leq w.$$

6 Compositionality

In this section, we start by defining the annotated types of our logic. The notion of annotated types is central in our work, as our model checking algorithm computes type-preserving representatives. The main technical contribution of this section is the proof of a "Feferman-Vaught style" composition lemma (Lemma 19), which is the main ingredient of the proof of correctness of our algorithm.

As reflected in the statement of this composition lemma, the notion of (plain) annotated types is not enough. What we show is that, in order to preserve the annotated type of a graph when replacing some part (of small interface) with a type-preserving representative, one needs to consider a stronger notion of representative, namely a representative of the same *extended type*. Extended types generalize types the same way that extended folios generalize folios and are also defined in this section.

Let us note that the employment of the extended type (instead of just the type) is required in order to retain control on the way that paths may cross the interface between the two glued graphs and avoid blow-ups in the involved constants. This phenomenon already appears in the compositionality of $\mathsf{FO{+}dp}$ [57] and is inspired by the corresponding statement for folios [37, Lemma 2.4].

Our proof follows the standard approach for MSO (see [35, Lemma 2.3]). However, one needs to show that when gluing along a small interface both the disjoint-paths predicate and the set quantification of our logic composes nicely (i.e., without blowups). For the disjoint-paths predicate, this was proven in [57], while for the set quantification, we show it here using Lemma 17.

Annotated types of our logic. Let $d, r, w \in \mathbb{N}$. For a given colored graph G and a sequence $\bar{R} = (R_1, \ldots, R_d)$ of subsets of $V(G)$, we define the *annotated* (d, r, w)*-type* of $(G, \bar{R})$ with respect to CMSO/ttw+dp, denoted by $\mathsf{type}_{d,r,w}(G, \bar{R})$, to be the set of all (up to logical equivalence) sentences φ of CMSO/ttw+dp (over the appropriate colored graph signature) in prenex normal form that have quantifier rank at most d, dp-rank at most r, and ttw-rank at most w and are satisfied in G when the ith variable of φ is interpreted in R_i, i.e., if the ith variable is a set variable, then it should be interpreted as a subset of R_i, while if it is a first-order variable it should be interpreted as an element of R_i. We remark that when we refer to the annotated type of an ℓ-boundaried graph, we assume the existence of ℓ colors in our signature and a bijection between the boundary vertices and the color each belongs to.

Extended annotated types. Recall that, given a boundaried graph $\mathbf{G} = (G, B, \rho)$ and $I \subseteq \binom{[|B|]}{2}$, we write $\mathbf{G}^I$ for the boundaried graph obtained from $\mathbf{G}$ by adding in its underlying graph the edges in $\{\{\rho^{-1}(i), \rho^{-1}(j)\} \mid \{i, j\} \in I\}$.

Let $d, r, w \in \mathbb{N}$. Given a boundaried graph $\mathbf{G} = (G, B, \rho)$ and a d-tuple $\bar{R}$ of subsets of $V(G)$, we define the *extended annotated* (d, r, w)*-type* of $(\mathbf{G}, \bar{R})$ to be

$$\mathsf{ext\text{-}type}_{d,r,w}(\mathbf{G}, \bar{R}) \quad = \quad \{(I, \mathsf{type}_{d,r,w}(\mathbf{G}^I, \bar{R})) \mid I \in \binom{[|B|]}{2}\}. \tag{2}$$

▶ Remark 18 (Sufficiency of rank bounds). In the following, whenever we speak of annotated (d, r, w)-types, we implicitly assume that the parameters d and r are chosen large enough as a function of w and the boundary size $|B|$. In particular, d and r are sufficient to encode all finite obstruction information required to test the constraint $\mathsf{p}(\cdot, \cdot) \leq k$ for every $k \leq w$ via extended folios, as guaranteed by Lemma 17. In other words, we assume that d and r are large enough, so that if two annotated boundaried graphs have the same extended annotated (d, r, w)-type, then they are $f_{17}^{\mathsf{ttw}}(w)$-equivalent, where f_{17}^{ttw} is the function of Lemma 17 for $\mathsf{p} = \mathsf{ttw}$.

Before proceeding to the proof of our "Feferman-Vaught style" composition lemma, we show how our definitions of compatibility and gluing generalize to boundaried graphs of the form $(\mathbf{G}, \bar{R})$. Let $\mathbf{G}_1 = (G_1, B_1, \rho_1)$ and $\mathbf{G}_2 = (G_2, B_2, \rho_2)$ be two boundaried graphs. Also, for given $d \in \mathbb{N}$, let $\bar{R}_1 := (R_1^1, \ldots, R_d^1)$ and $\bar{R}_2 := (R_1^2, \ldots, R_d^2)$ be two d-tuples of subsets of $V(G_1)$ and $V(G_2)$, respectively. We say that $(\mathbf{G}_1, \bar{R}_1)$ and $(\mathbf{G}_2, \bar{R}_2)$ are *compatible* if $\rho_2^{-1} \circ \rho_1$ is an isomorphism from $G_1[B_1]$ to $G_2[B_2]$ such that for every $i \in [d]$, $\rho_1(R_i^1 \cap B_1) = \rho_2(R_i^2 \cap B_2)$. Given that $(\mathbf{G}_1, \bar{R}_1)$ and $(\mathbf{G}_2, \bar{R}_2)$ are compatible, we define $(\mathbf{G}_1, \bar{R}_1) \oplus (\mathbf{G}_2, \bar{R}_2)$ as the pair $(G, \bar{R})$, where

- G is the graph obtained if we take the disjoint union of G_1 and G_2 and, for every $i \in [|B_1|]$, we identify the vertices $\rho_1^{-1}(i)$ and $\rho_2^{-1}(i)$, agreeing that after each such identification the vertices of the boundary of $\mathbf{G}_1$ prevail, and
- $\bar{R}$ is the d-tuple $(R_1, \ldots, R_d)$, where for each $i \in [d]$, R_i is obtained by the disjoint union of R_i^1 and R_i^2 and, for every $i \in [|B_1|]$, if $\rho_1^{-1}(i) \in R_i^1$, we identify the vertices $\rho_1^{-1}(i)$ and $\rho_2^{-1}(i)$, agreeing that after each such identification the vertices of $R_i^1 \cap B_1$ prevail.

We are now in position to state the following compositionality lemma, whose proof uses Lemma 17.

▶ **Lemma 19.** *Let $d, r, w \in \mathbb{N}$, let $\mathbf{G}_1$, $\mathbf{G}_2$ be two boundaried graphs and let $\bar{R}_1$,$\bar{R}_2$ be two d-tuples of subsets of $V(G_1)$ and $V(G_2)$, respectively. If*

- *$(\mathbf{G}_1, \bar{R}_1)$ and $(\mathbf{G}_2, \bar{R}_2)$ are compatible, and*
- *the extended annotated (d, r, w)-type of $(\mathbf{G}_1, \bar{R}_1)$ and $(\mathbf{G}_2, \bar{R}_2)$ is the same,*

then for every boundaried graph $\mathbf{C}$ *and every* d*-tuple* $\bar{Q}$ *of subsets of* $V(C)$ *such that* $(\mathbf{C}, \bar{Q})$ *is compatible with* $(\mathbf{G}_1, \bar{R}_1)$ *(and* $(\mathbf{G}_2, \bar{R}_2)$*), we have that*

$(\mathbf{C}, \bar{Q}) \oplus (\mathbf{G}_1, \bar{R}_1)$ *and* $(\mathbf{C}, \bar{Q}) \oplus (\mathbf{G}_2, \bar{R}_2)$ *have same annotated* (d, r, w)*-type.*

7 Computing representatives

Our dynamic programming algorithm works bottom-up over an unbreakable decomposition and computes a representative of each bag. For a node t of a tree decomposition, we use $\mathbf{G}_t$ to denote the boundaried graph $(G[\mathsf{cone}(t)], \mathsf{adh}(t), \rho)$, for some $\rho \colon \mathsf{adh}(t) \to |\mathsf{adh}(t)|$.

(d, r, w)-representatives. Let $d, r, w \in \mathbb{N}$. Let $\mathbf{G}, \mathbf{G}'$ be two boundaried graphs and let $\bar{R}$ and $\bar{R}'$ be two d-tuples of subsets of $V(G)$ and $V(G')$, respectively. We say that $(\mathbf{G}', \bar{R}')$ is a (d, r, w)-*representative* of $(\mathbf{G}, \bar{R})$ if $\mathsf{ext\text{-}type}_{d,r,w}(\mathbf{G}, \bar{R}) = \mathsf{ext\text{-}type}_{d,r,w}(\mathbf{G}', \bar{R}')$.

The main subroutine of our algorithm is given in the following lemma. It uses Lemma 12 to show that we can rewrite every given sentence of CMSO/ttw+dp to a sentence of FO+dp, assuming that we work with unbreakable graphs. After this rewriting (with some controlled blow-up in the quantifier rank, but not in the dp-rank), we can use the model checking algorithm for FO+dp [57].

▶ **Lemma 20.** *There is a function* $f_{\mathsf{size}} \colon \mathbb{N}^4 \to \mathbb{N}$ *and an algorithm that, given:*
- *integers* $\ell, q, d, r, w \in \mathbb{N}$*;*
- *a graph* G *that excludes a graph* H *as a topological minor;*
- *a* d*-tuple* $\bar{R}$ *of subsets of* $V(G)$*;*
- *a tree decomposition* (T, bag) *of* G *of adhesion* ℓ*; and*
- *a node* t *of* T *such that* $G[\mathsf{cone}(t)]$ *is* $(q, w+1)$*-unbreakable;*

outputs, in time $\mathcal{O}_{|H|,r,\ell,q,d,w}(|\mathsf{cone}(t)|^3)$*, a* (d, r, w)*-representative of* $(\mathbf{G}_t, \bar{R} \cap \mathsf{cone}(t))$ *of size at most* $f_{\mathsf{size}}(d, r, w, \ell)$*.*

Note that Lemma 20 requires the whole cone under consideration to be unbreakable. We show that unbreakability (with worse bounds) is preserved when the subgraphs corresponding to subtrees of the current node have bounded size.

8 The model checking algorithm

In this section, we present our model checking algorithm, i.e., the proof of Theorem 3. Before presenting the algorithm, we show that when replacing a part of the graph by a representative, the combinatorial assumptions on the graph and the tree decomposition are preserved. This is used to show the correctness of our dynamic-programming procedure.

8.1 Preserving properties after gadget replacement

In order to show that the combinatorial assumptions on the graph and the tree decomposition are preserved, we need to derive certain properties from the extended annotated type of a given boundaried graph. In particular, it is easy to observe that for sufficiently large values of d and r (depending on δ and ℓ), the extended annotated (d, r, w)-type of an ℓ-boundaried graph $\mathbf{G}$ can encode the graph induced by $B(\mathbf{G})$, the extended δ-folio of $\mathbf{G}$, and whether $G \setminus B(\mathbf{G})$ is connected. This follows from the fact that all the above can be expressed in FO+dp and is materialized in the following statement.

▶ **Observation 21.** *There are two functions $g, g'\colon \mathbb{N}^2 \to \mathbb{N}$ such that the following holds. Let $\ell, \delta, w \in \mathbb{N}$ and set $d := g(\ell, \delta)$ and $r := g'(\ell, \delta)$. Let $\mathbf{G}, \mathbf{G}'$ be two ℓ-boundaried graphs and let $\bar{R}$ and $\bar{R}'$ be two d-tuples of subsets of $V(G)$ and $V(G')$, respectively. If $\mathsf{ext\text{-}type}_{d,r,w}(\mathbf{G}, \bar{R}) = \mathsf{ext\text{-}type}_{d,r,w}(\mathbf{G}', \bar{R}')$, then*

- *$(\mathbf{G}', \bar{R}')$ is compatible with $(\mathbf{G}, \bar{R})$;*
- *the extended δ-folios of $\mathbf{G}$ and $\mathbf{G}'$ are the same; and*
- *$G' \setminus B(\mathbf{G}')$ is connected if and only if $G \setminus B(\mathbf{G})$ is connected.*

Preserving regularity and unbreakability. Recall that, given a graph G, a tree decomposition $\mathcal{T} = (T, \mathsf{bag})$, and a node $t \in V(T)$, $\mathbf{G}_t$ denotes the boundaried graph $(G[\mathsf{cone}(t)], \mathsf{adh}(t), \rho)$, for some $\rho\colon \mathsf{adh}(t) \to |\mathsf{adh}(t)|$. We also use G_t to denote the graph $G[\mathsf{cone}(t)]$. We next show that in a tree decomposition of adhesion ℓ, if we replace G_t with a graph with the same ℓ-folio that also preserves the connectivity of the component, we have that regularity and strong unbreakability are preserved in the obtained tree decomposition.

▶ **Lemma 22.** *Let $\ell \in \mathbb{N}$. Let G be a graph and let $\mathcal{T} = (T, \mathsf{bag})$ be a tree decomposition of G of adhesion ℓ. Let t be a node in $V(T)$. Let $\mathbf{G}'$ be a boundaried graph such that $\mathbf{G}_t$ and $\mathbf{G}'$ have the same ℓ-folio and $G' \setminus B(\mathbf{G}')$ is connected if and only if $G_t \setminus B(\mathbf{G}_t)$ is connected. Also, let $\hat{G}$ be the graph obtained from G by replacing $G[\mathsf{cone}(t)]$ with G'. Let $\hat{\mathcal{T}}$ be the tree decomposition of $\hat{G}$ obtained by $\mathcal{T}$ after removing all (strict) descendants of t and setting $\mathsf{bag}(t) = V(G')$. Then, the following properties hold:*

- *$\hat{\mathcal{T}}$ has the same adhesion as $\mathcal{T}$;*
- *if $\mathcal{T}$ is regular, then $\hat{\mathcal{T}}$ is also regular; and*
- *for every node $x \neq t$ of $\hat{\mathcal{T}}$, it holds that if x has the strong (q,k)-unbreakability property in $\mathcal{T}$, then it also has it in $\hat{\mathcal{T}}$.*

Preserving topological-minor-freeness. We also need to show that when replacing with a representative of the same extended folio, topological-minor-freeness is preserved. A similar version of this result was shown in [57].

▶ **Lemma 23.** *Let $c, h \in \mathbb{N}$. Let G be a graph and let $\mathcal{T}$ be a tree decomposition of G. Let t be a non-root node in $V(T)$. Let $\mathbf{G}'$ be a boundaried graph of order at most c such that $\mathbf{G}_t$ and $\mathbf{G}'$ are h-equivalent and let $\hat{G}$ be the graph obtained from G by replacing $G[\mathsf{cone}(t)]$ with G'. Then, if G excludes a graph H of size at most h as a topological minor, then the same holds for $\hat{G}$.*

8.2 Dynamic programming over unbreakable decompositions – Proof of Theorem 3

Assume that we are given a formula φ of our logic in prenex normal form with quantifier rank d_0, dp-rank r_0, and ttw-rank w and a graph G that excludes a graph H as a topological minor. We set $k := w + 1$, $\ell := q(k)$, $q := q(k)$, where $q(\cdot)$ is the function of Proposition 8, and $\delta = \max\{\ell, |H|\}$. We also set $d = \max\{d_0, g(\ell, \delta)\}$ and $r = \max\{r_0, g'(\ell, \delta)\}$, where g and g' are the functions of Observation 21.

The algorithm of Theorem 3 works as follows. We first compute a regular strongly (q,k)-unbreakable decomposition $\mathcal{T} = (T, \mathsf{bag})$ of adhesion at most ℓ using the algorithm of Proposition 8. Then, we consider a post-order traversal of T, i.e., an ordering of $V(T)$ such that each node is visited after all its (strict) descendants are visited. We process the nodes by increasing order. Before any iteration, we set $G^\bullet := G$, $\bar{R} = V(G)^d$, and $\mathcal{T}^\bullet := \mathcal{T}$.

Iteration step. We work with the graph $G^\bullet$ and its tree decomposition $\mathcal{T}^\bullet$. Let t be the node of $\mathcal{T}^\bullet$ that is currently processed.

Let $\mathsf{MaxAdh}(t)$ be the set of all children z of t such that there is no child $z' \neq z$ of t such that $\mathsf{adh}(z) \subseteq \mathsf{adh}(z')$. Note that for every child y of t there is a $z \in \mathsf{MaxAdh}(t)$ such that $\mathsf{adh}(y) \subseteq \mathsf{adh}(z)$. For every $z \in \mathsf{MaxAdh}(t)$, we set

$$G_z^{\mathsf{Max}} := \bigcup \{G^\bullet[\mathsf{cone}(y)] \mid y \text{ is a child of } t \text{ with } \mathsf{adh}(y) \subseteq \mathsf{adh}(z)\}$$

and $\mathbf{G}_z^{\mathsf{Max}} := (G_z^{\mathsf{Max}}, \mathsf{adh}(z), \rho)$, where ρ is an arbitrary bijection from $\mathsf{adh}(z)$ to $|\mathsf{adh}(z)|$.

Let $(G', \bar{R}')$ be the (annotated) graph obtained from $(G^\bullet, \bar{R}^\bullet)$ by replacing, for each $z \in \mathsf{MaxAdh}(t)$, $(\mathbf{G}_z^{\mathsf{Max}}, \bar{R}^\bullet \cap V(G_z^{\mathsf{Max}}))$ with $(\mathbf{H}_z, \bar{R}_z)$. Also, let $\mathcal{T}'$ be the tree decomposition obtained from $\mathcal{T}^\bullet$ after removing every node in the subtree rooted at t, except of the nodes in $\mathsf{MaxAdh}(t)$ and by setting, for each $z \in \mathsf{MaxAdh}(t)$, $\mathsf{bag}(z) = V(H_z)$.

We set

- $\mathbf{C} := (G' \setminus \mathsf{comp}(t), \mathsf{adh}(t), \rho)$, where ρ is an arbitrary bijection from $\mathsf{adh}(t)$ to $|\mathsf{adh}(t)|$;
- $\bar{Q} := \bar{R}' \setminus \mathsf{comp}(t)$;
- $\mathbf{G}'_t := (G'_t, \mathsf{adh}(t), \rho)$, where G'_t is the subgraph of G' induced by the cone of t in $\mathcal{T}'$; and
- $\bar{R}'_t := \bar{R}' \cap V(G'_t)$.

Then, we apply the algorithm of Lemma 20 for the graph G', the tree decomposition $\mathcal{T}'$, and the node t. If it outputs a (d, r, w)-representative $(\mathbf{G}^\star, \bar{R}^\star)$ of $(\mathbf{G}'_t, \bar{R}')$, then we set $(G^\bullet, \bar{R}^\bullet) := (\mathbf{C}, \bar{Q}) \oplus (\mathbf{G}^\star, \bar{R}^\star)$. Also, set $\mathcal{T}^\bullet$ to be the tree decomposition obtained from $\mathcal{T}'$ after removing all children of t. This finishes the iterative step.

Final step. When all nodes have been processed, apply Courcelle's theorem [13] to decide if $G^\bullet$ satisfies φ when its quantified variables are interpreted in the corresponding sets $R_i^\bullet$, and report the corresponding answer.

The details regarding the correctness of the above algorithm and its claimed running time are deferred to the full version.

9 Expressive power

In the full version of the paper, we discuss the expressive power of the logics CMSO/ttw and CMSO/ttw+dp compared to other logics mentioned in this paper (e.g. FO+conn, FO+dp, CMSO). Due to space constraints, we only present the results in Figure 2.

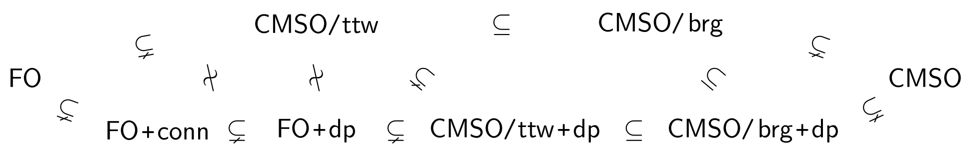

Figure 2 Comparison of the logics.

10 Hardness on monotone classes that do not exclude a topological minor

As mentioned in the introduction, on monotone classes of graphs not excluding any graph as a topological minor, hardness of model checking for CMSO/ttw+dp follows directly from hardness of FO+conn [49], since FO+conn is contained in CMSO/ttw+dp. It is therefore natural to ask whether, in the absence of the disjoint-paths predicate, i.e., in the fragment CMSO/ttw, the hardness still holds or it is possible to have efficient model checking for this fragment beyond topological-minor-free classes. In this section, we prove that it is not the case, namely by showing that CMSO/ttw can encode all graphs in any monotone class of graphs that does not exclude a fixed graph as a topological minor. The key observation is that such a class contains an induced subdivision of every graph, and that these subdivisions can be undone with an CMSO/ttw-formula. If the class additionally admits efficient encoding, then we can derive hardness of model checking for CMSO/ttw.

▶ **Definition 24.** *A graph class $\mathscr{C}$* admits efficient encoding of topological minors *if for every graph H, there exists a graph $G \in \mathscr{C}$ such that H is a topological minor of G, and moreover, given H, one can compute such a graph G together with an explicit topological-minor model of H in G in time polynomial in $|H|$.*

▶ **Theorem 25.** *Let $\mathscr{C}$ be a monotone graph class that admits efficient encoding of topological minors. Then* CMSO/ttw *model checking on $\mathscr{C}$, parameterized by the formula size, is $AW[*]$-hard.*

Note that the proof of Theorem 25 holds not only for CMSO/ttw, but for CMSO/p for any parameter p that is the *torso extension* (defined in Section 11) of a parameter that is bounded on the class of cycles, such as treewidth or pathwidth (but not treedepth).

11 Research directions and open questions

In this paper, we studied two fragments of CMSO, namely CMSO/ttw+dp and CMSO/atw+dp, generated by the annotated graph parameters ttw and atw (equivalently, bog and brg), which represent two distinct parametric extensions of treewidth. On H-topological-minor-free graph classes, we establish an algorithmic meta-theorem for CMSO/ttw+dp, complementing the known results of [55] for CMSO/atw+dp on H-minor-free classes. Together, these results support our view that *annotated parameters* provide a robust interface between logic and structure, as they nicely describe how complicated the quantified sets may be, while still being rich enough to cover many algorithmic applications.

A natural next step is to broaden the framework beyond ttw and atw. One systematic way to do this starts from an arbitrary minor-monotone graph parameter p and derives two canonical annotated variants. We define the *annotated extension* of p, denoted by ap, by letting $\mathsf{ap}(G, X)$ be the maximum value of p over all X-minors of G. We also define the *torso extension* of p, denoted by tp, by letting $\mathsf{tp}(G, X)$ be the minimum value of $\mathsf{p}(\mathsf{torso}(G, X'))$ over all $X' \supseteq X$. These two constructions are closely related but, in general, need not coincide. Understanding when they do coincide is a central structural question (see [50]).

We conclude with a small list of concrete questions that we find particularly compelling at the moment.

1. **Model checking CMSO/atw+dp on topological-minor-free classes.** Can we extend the tractability of CMSO/atw+dp from minor-free classes [55] to topological-minor-free classes? We are not in position to conjecture whether our approach for model checking in the present paper can work also for CMSO/atw+dp or that minor-freeness is the tractability boundary for this logic.
2. **Equivalence under functional equivalence of parameters.** Suppose $\mathsf{p} \sim \mathsf{p}'$ (in the sense used in Lemma 9). Do the fragments CMSO/p and CMSO/p′ already coincide in expressive power *without* the dp-predicate? Establishing this would clarify how much of the robustness of Lemma 9 relies on the presence of dp.

3. **Model checking beyond topological-minor-free classes for CMSO/td.** What is the precise tractability boundary for model checking CMSO/td? In particular, does fixed-parameter tractability extend meaningfully beyond topological-minor-free classes, or is there a natural hardness threshold?

4. **Improving the running time of the meta-theorem.** Our current running time is dominated by the computation of an unbreakable decomposition, which is the main obstacle to getting below cubic dependence in the input size. Can this bottleneck be removed or replaced by a faster decomposition scheme? Similar considerations apply to the best known implementations of FO+dp.

5. **A dense analogue for hereditary classes.** Can one develop an analogue of the present framework for hereditary (possibly dense) graph classes? This likely requires annotated parameters that are *not* minor-monotone, but still support a replacement/compositionality theory strong enough to yield algorithmic meta-theorems.

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