Axiomatizability of Alexandrov Dynamic Topological Logic

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Abstract

Dynamical systems provide rigorous models of movement or evolution over time. Due to their abstract nature, they may be naturally employed for representing e.g. physical, biological, or financial phenomena. Specifically in the context of Computer Science, computational processes, machine learning algorithms, and multi-agent systems may be regarded as dynamical systems. This has sparked interest in designing formal specification languages for dynamical systems which could potentially be employed for automated or computer-assisted deduction, leading to the introduction of *dynamic topological logic* (DTL). When space is continuous but time is discrete, it is known that a sound and complete deductive calculus for DTL exists. However, discrete spaces are not uncommon in CS applications, and in this setting, whether such a calculus exists even in principle has been an open question for more than two decades. More precisely, it was unknown whether the DTL of Alexandrov spaces is computably enumerable. In this paper, we use model search techniques to provide an affirmative answer.

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1 Introduction

The dynamical topological logic project [3, 28] aimed to develop specification languages for dynamical systems expressive enough to capture complex properties of their behaviour whilst retaining computational properties desirable for computer-assisted or automated proof. Interest in specialized logical systems for dynamical systems stems from their wide potential for applicability in science and technology and mirrors efforts in AI to develop efficient methods for reasoning about space and time; see [7, 38] for overviews of qualitative spatial reasoning. Modern applications of qualitative spatial reasoning include geolocalisation, for which the *region connection calculus* [12, 34], was introduced, and self-driving cars, requiring the combination of spatial and temporal reasoning [10, 25].

Dynamic topological logic (DTL) is a propositional modal logic which involves the modality $\Diamond$ ("nearby" or "closure") and the tenses $\bullet$ ("next") and $\blacklozenge$ ("eventually"). This language is interpreted over *dynamical systems*, which for our purpose, consist of a tuple $\langle \mathfrak{X}, f \rangle$, where $\mathfrak{X}$ is a topological space and f a continuous function from $\mathfrak{X}$ to $\mathfrak{X}$. The space

$\mathfrak{X}$ is used for interpreting $\Diamond$ and each application of f represents a "tick of the clock" when interpreting tenses. Within this formal language, we can express properties such as "there are points nearby" ($\Diamond p$), "after one tick of the clock, the region p is reached" ($\bullet p$), or "after some (unspecified) amount of time, the region p will be reached" ($\blacklozenge p$).[1] The purely topological fragment (i.e., with $\Diamond$ alone) is related to the region connection calculus RCC8; specifically, RCC8 embeds into its extension with the universal modality [40].[2]

More complex properties may be represented by combining modalities. For example, $\Diamond\bullet p \to \bullet\Diamond p$ states that nearby points will stay nearby after one tick of the clock or, more formally, that f is continuous. Accordingly, the formula

$$\texttt{Alex} \coloneqq \Diamond\blacklozenge p \to \blacklozenge\Diamond p$$

states that nearby points will "always remain near", a property that is characteristic of Alexandrov spaces, topological spaces whose open sets are the upsets of a fixed preorder $\sqsubseteq$. Unlike in the general topological case, Alexandrov spaces may be represented as perhaps more familiar Kripke structures. They moreover allow for a very simple characterization of continuity: a function f is continuous if and only if $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$. This further implies $f^n(x) \sqsubseteq f^n(y)$ for all $n \in \mathbb{N}$, from which Alex may be derived, despite not being valid over topological spaces in general, see Example 2.10.

There are several reasons why the restriction to Alexandrov spaces is of interest. It is not uncommon for topological spaces arising in CS to be Alexandrov; for example, every finite space is of this kind. They also provide a bridge between DTL and other topics in computational logic, allowing for the transfer of techniques and results. The semantics of Alexandrov DTL is closely related to first-order temporal logics in which each moment in time is assigned a first-order structure (see e.g. [29]). Specifically, Alexandrov DTL may be regarded as a fragment of this logic in the setting where the first-order domains are preorders which expand over time. Alternately, Alexandrov DTL may be identified with the expanding product of linear temporal logic with S4 [24]. Expanding products of modal logics skirt the boundary between decidability and undecidability (or even non-axiomatizability) with many challenging questions long remaining open. Focusing on Alexandrov spaces thus broadens the applicability of our work beyond the realm of topological dynamics.

The "near-next" (or "closure-next") fragment of DTL is rather well behaved, with a decidable satisfiability problem which does not distinguish Alexandrov spaces from general topological ones [3, 28]. In contrast, logics with $\blacklozenge$ tend to be undecidable [26] or even non-axiomatizable when f is a homeomorphism [27]. But when f is only continuous, DTL is axiomatizable and enjoys a Hilbert calculus in an extended language [14, 19]. Note that this result applies only to general DTL, which is distinct from Alexandrov DTL [28] (see Example 2.10). The latter is known only to be undecidable [26], with axiomatizability remaining a longstanding open problem.

In this paper, we settle this question and show that Alexandrov DTL is indeed axiomatizable. We follow a technique based on Kruskal's Tree Theorem [30], applied to various modal logics by Gabelaia et al. [24], and adapted to DTL by Konev et al. [26], who showed that the fragment DTL_1, in which $\blacklozenge$ does not appear in the scope of $\Diamond$, is computably enumerable. Fernández-Duque [14] extended this result to the full language. However, while DTL_1 over Alexandrov spaces coincides with that over arbitrary topological spaces, this is

[1] To be precise, all of these properties are stated relative to an evaluation point x, so that e.g. "nearby" should be understood as "arbitrarily close" (in the topological sense), and by "reached" we mean that $f^n(x)$ will belong to the region p for some $n \geq 0$.

[2] We do not include the universal modality in our formal language, but our proofs can be modified to accommodate it as in e.g. [15].

not the case for the full language, and Fernández-Duque's [14] proof applies specifically to the general topological case. We close this gap by showing that DTL of Alexandrov spaces is also computably enumerable.

The advantage of the sub-language DTL_1 is that a formula is satisfiable if and only if it is satisfied on a *locally finite space*, which in turn may be represented as a disjoint union of finite preorders we call *quasi-states*. This allows one to perform a model search procedure, with "loops" recognized by embedding of subsequent quasi-states. Unsuccessful search is guaranteed to eventually loop by Kruskal's Theorem.

The main challenge in adapting this to the full language is that models may no longer be represented as a disjoint union of finite quasi-states. This is circumvented by replacing the transition function f by a non-deterministic relation, which can later be "unwound" to produce a deterministic model which might not be locally finite. The technique has since been adapted to various related logics, including some non-classical temporal logics [1, 20].

Oddly enough, this challenge persists in the Alexandrov setting, since while structures may readily be represented as posets, these are in general not locally finite. As in the general topological case, we remedy this by replacing f by a non-deterministic relation and then applying an unwinding. The crucial difference is that f must already be deterministic on increasingly large segments of each quasi-state in order to ensure that the unwound model is preordered and hence Alexandrov. Thus, our proof builds on established techniques, but modifies and expands on them in an essential way to finally give a positive answer to the question of axiomatizability for Alexandrov DTL.

2 Preliminaries

In this section we review the syntax and semantics of dynamic topological logic. On the semantic front, we briefly recall the topological semantics, but then focus mainly on the relational or Kripke semantics, which is equivalent to the topological one for Alexandrov spaces.

Syntax

Fix a countable set $\mathbb{P}$ of *atomic propositions*. The set of *DTL-formulas* Form, or *formulas* for short, is defined by the Backus–Naur form

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \bullet\varphi \mid \blacklozenge\varphi,$$

where $p \in \mathbb{P}$. Define shorthands $\vee$ and $\to$ in the standard way, e.g. $\varphi \to \psi := \psi \vee \neg\varphi$. The modal operators $\bullet$ and $\blacklozenge$ are intended to be interpreted as LTL-like *next* (X) and *eventually* (F) respectively, and called *temporal*, while $\Diamond$ is intended to express a modality local to a system state at a certain point in time, and will be interpreted topologically or relationally. We call DTL-formulas that are free of the temporal modalities $\bullet$ and $\blacklozenge$ *unimodal formulas*.

Topological Semantics

We briefly recall the basics of topology necessary for the topological semantics of DTL.

▶ **Definition 2.1** (Topological space). *A* topological space *is a pair $\langle X, \tau \rangle$ where X is a set and $\tau \subseteq \mathcal{P}(X)$, such that*
- $X \in \tau$,
- *τ is closed under binary intersections, and*
- *τ is closed under arbitrary unions.*

The space is called Alexandrov *iff τ is also closed under arbitrary (non-empty) intersections. The set τ is called a* topology on X*, its elements are called* open*, and their complements* closed*.*

Topological spaces arise across mathematics and computer science, from mathematical analysis to topological data analysis [11] and denotational semantics for programming languages [35, 36], to model shapes, continuity and infinite processes.

▶ **Example 2.2** (Euclidean topology). Let $\mathbb{R}$ denote the set of real numbers. The collection of all sets $A \subseteq \mathbb{R}$ that can be written as unions of open intervals (a, b) forms a topology on $\mathbb{R}$, known as the *Euclidean topology.*

Alexandrov spaces are exactly the spaces for which the collection of closed sets forms a topology, albeit in general one distinct from the topology of the space. As such, Alexandrov spaces come in dual pairs. A further useful property is that every Alexandrov space arises from a preorder, as in the following example. This observation is key to the equivalence between topological and relational semantics for modal logics like DTL.

▶ **Example 2.3.** Let X be some set and $\sqsubseteq$ a preorder, i.e. a reflexive and transitive relation, on X. Then the collection of all upward closed sets of X forms an Alexandrov topology on X.

On a topological space, $\Diamond$ is interpreted as the *topological closure* operation. For obvious reasons, this is called *closure semantics*, and was first studied by McKinsey and Tarski [31].

▶ **Definition 2.4** (Topological closure). *Let $\langle X, \tau\rangle$ be a topological space and $A \subseteq X$. Then the* (topological) closure *of A is least closed set extending A:*

$$\mathrm{cl}(A) := \bigcap \{C \in \mathcal{P}(X) \mid C \supseteq A, C \text{ closed}\}.$$

The $\bullet$ modality is interpreted by a *continuous* function on the space.

▶ **Definition 2.5** (Continuous function). *Let $\langle X, \tau\rangle$ and $\langle Y, \sigma\rangle$ be topological spaces and $f : X \to Y$. Then f is called* continuous *from $\langle X, \tau\rangle$ to $\langle Y, \sigma\rangle$ iff*

$$\forall A \in \sigma.\ f^{-1}(A) \in \tau,$$

where $f^{-1}(A) := \{x \in X \mid f(x) \in A\}$ denotes the set of all f-preimages of A.

Formulas are now interpreted in topological spaces equipped with a continuous endofunction and a valuation map.

▶ **Definition 2.6** (Dynamic topological model). *A* dynamic topological model*, or* DTM *for short, is a tuple $\langle \mathfrak{X}, f, V\rangle$ where $\mathfrak{X} = \langle X, \tau\rangle$ is a topological space, f a continuous function from $\mathfrak{X}$ to $\mathfrak{X}$ and $V : \mathbb{P} \to \mathcal{P}(X)$. The interpretation of formulas is defined recursively by:*

$$\begin{aligned} [\![p]\!] &:= V(p) & [\![\Diamond\varphi]\!] &:= \mathrm{cl}([\![\varphi]\!]) \\ [\![\varphi \wedge \psi]\!] &:= [\![\varphi]\!] \cap [\![\psi]\!] & [\![\bullet\varphi]\!] &:= f^{-1}([\![\varphi]\!]) \\ [\![\neg\varphi]\!] &:= X \setminus [\![\varphi]\!] & [\![\blacklozenge\varphi]\!] &:= \bigcup \{[\![\bullet^n\varphi]\!] \mid n \in \mathbb{N}\}, \end{aligned}$$

where $\bullet^n$ denotes n repetitions of $\bullet$. We say that a formula φ is valid *on $\mathfrak{M}$ iff $[\![\varphi]\!] = X$.*

Now, *Alexandrov DTL* is the set of formulas valid on every DTM where the underlying topological space is Alexandrov. Due to a categorical isomorphism between Alexandrov spaces and preorders, which was already partially given in Example 2.3, the topological Alexandrov semantics is equivalent to a relational one using preorders, which we will describe now. Since preorders tend to be easier to work with combinatorially, we will work with the relational semantics in the rest of the paper, and the results translate back to the topological Alexandrov setting.

Relational Semantics

Let X be a set and $\sqsubseteq$ be a preorder on it. We use the term preorder for both the relation $\sqsubseteq$ and the pair $\langle X, \sqsubseteq \rangle$ consisting of the set and the relation . In modal logic, the latter is known as a (Kripke) frame, and provides an interpretation for the modality $\Diamond$ by defining, for $A \subseteq X$,

$$\Diamond A \coloneqq \{x \in X \mid \exists y \in A.\ x \sqsubseteq y\}. \tag{1}$$

This set is also called the *downset generated* by A, since it is the least downward closed set extending A.[3] *Generated upsets* are defined dually, and we say A *generates* $\langle X, \sqsubseteq \rangle$ iff the upset generated by A equals X.

Unimodal formulas are interpreted on preorders with a valuation.

▶ **Definition 2.7** (Preorder model). *A* reflexive transitive Kripke model *or* preorder model *is a tuple $\mathfrak{M} = \langle |\mathfrak{M}|, \sqsubseteq_{\mathfrak{M}}, V_{\mathfrak{M}} \rangle$ where $\sqsubseteq_{\mathfrak{M}}$ is a preorder on $|\mathfrak{M}|$, and $V_{\mathfrak{M}} \colon \mathbb{P} \to \mathcal{P}(|\mathfrak{M}|)$. The interpretation of unimodal formulas is defined by induction over formulas:*

$$\llbracket p \rrbracket \coloneqq V(p) \qquad \llbracket \neg\varphi \rrbracket \coloneqq X \setminus \llbracket \varphi \rrbracket$$
$$\llbracket \varphi \wedge \psi \rrbracket \coloneqq \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \qquad \llbracket \Diamond\varphi \rrbracket \coloneqq \Diamond\llbracket \varphi \rrbracket,$$

where $\Diamond$ on a set is defined as in eq. (1). We say that $\mathfrak{M}$ satisfies *a unimodal formula φ iff there exists $x \in |\mathfrak{M}|$ such that $x \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$, in which case we also say x* satisfies *φ. Validity is defined analogous to Definition 2.6.*

For $x, y \in X$, we say
- y is a *successor* of x iff $x \sqsubseteq y$,
- y is a *strict successor* of x iff $x \sqsubseteq y$ and $x \not\sqsupseteq y$,
- $x \equiv y$ iff $x \sqsubseteq y$ and $y \sqsubseteq x$, and
- y is an *immediate successor* of x iff it is a strict successor and for every $z \in X$ with $x \sqsubseteq z \sqsubseteq y$, either $z \equiv x$ or $z \equiv y$.

The equivalence classes of $\equiv$ are called *clusters*.

As in the topological semantics, $\bullet$ is interpreted by a function, in this case an order-preserving or *monotone* one. We define this property generally for relations, where we call it *continuity* instead, and view functions as a special case of a relation.

[3] Note the analogy with the topological semantics, where $\Diamond$ is interpreted by the topological closure. Indeed, in Example 2.3, the topologically closed sets are exactly the downward closed ones, and the two interpretations of $\Diamond$ coincide.

$$\begin{tikzcd} x' \arrow[r, dashed, "R"] & \exists y' \\ x \arrow[u, "\sqsubseteq_X"] \arrow[r, "R"'] & y \arrow[u, dashed, "\sqsubseteq_Y"'] \end{tikzcd}$$

Figure 1 Diagrammatic representation of the continuity condition for relations between preorders, as defined in Definition 2.8.

▶ **Definition 2.8** (Continuity, monotonicity). *Let $\langle X, \sqsubseteq_X \rangle$ and $\langle Y, \sqsubseteq_Y \rangle$ be preorders and $R \subseteq X \times Y$. Then R is called* continuous *(also* forward confluent*) iff for all $x, x' \in X$ and $y \in Y$ with $x \sqsubseteq_X x'$ and $\langle x, y \rangle \in R$ there exists $y' \in Y$ with $y \sqsubseteq_Y y'$ and $\langle x', y' \rangle \in R$. This requirement is depicted diagrammatically in Figure 1. If R is a function*[4] *then it is called* monotone *or* order preserving.

The relational semantics is now entirely analogous to the topological one.

▶ **Definition 2.9** (Dynamic preorder model). *A* dynamic preorder model*, or* DPM *for short, is a preorder model $\mathfrak{M}$ together with a monotone function $f_{\mathfrak{M}}$ from $\mathfrak{M}$ to $\mathfrak{M}$. For $x \in |\mathfrak{M}|$, the point $f_{\mathfrak{M}}(x)$ is called the* temporal successor *of x. The interpretation, validity and satisfaction of DTL-formulas are defined as in Definitions 2.6 and 2.7.*

DTL over preorders is the set of formulas valid on every DPM, but due to a categorical isomorphism mentioned earlier this equals Alexandrov DTL [2, Section 1][32, Proposition 2.4.6].

▶ **Example 2.10.** Recall from the introduction that we defined `Alex` to be $\Diamond\blacklozenge p \to \blacklozenge\Diamond p$. `Alex` is valid over the class of DPMs, but not over the class of DTMs [28]. To see the latter, consider a DTM $\langle \mathbb{R}, f, V \rangle$, where $f(x) = 2x$ and $V(p) = (1, \infty)$. Then, if $x > 0$, it is not hard to check that $f^n(x) \in V(p)$ for large enough n, so $x \in [\![\blacklozenge p]\!]$, and hence $0 \in [\![\Diamond\blacklozenge p]\!]$. Meanwhile, $f^n(0) = 0$ for all n, whence $0 \notin [\![\blacklozenge\Diamond p]\!]$ and $0 \notin [\![\texttt{Alex}]\!]$.

In contrast, if $\langle W, \sqsubseteq, f, V \rangle$ is a DPM and $w \in [\![\Diamond\blacklozenge p]\!]$, we may choose $v \sqsupseteq w$ such that $v \in [\![\blacklozenge p]\!]$ and hence $f^n(v) \in [\![p]\!]$ for some n. From this, it is easy to see using monotonicity of f that $f^n(w) \in [\![\Diamond p]\!]$ and hence $w \in [\![\blacklozenge\Diamond p]\!]$.

While the relational semantics is already easier to handle combinatorially and computationally than the topological semantics, it still does not enjoy the locally finite model property, as we will see in the next section. In section Sections 4 and 5, we overcome this with a third semantics.

3 Failure of the Locally Finite Model Property

It should come as no surprise that Alexandrov DTL fails to have the finite model property, given that it is known to be undecidable [26]; more specifically, the set of satisfiable formulas is not computably enumerable. It is perhaps more surprising that it also fails the locally finite model property, given that some closely related logics, such as some expanding products [23] and the DTL_1 fragment of DTL [26], do have this property. To be precise, the *locally finite model property fails* when the following holds.

[4] That is, for every $x \in X$, there exists a unique $y \in Y$ such that $\langle x, y \rangle \in R$.

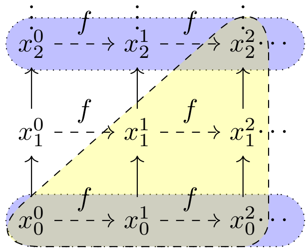

Figure 2 Model $\mathfrak{M}$ from Lemma 3.1. The atomic proposition p is satisfied in the even rows (blue, dotted), while q is satisfied in the lower triangle (yellow, dashed).

▶ **Lemma 3.1.** *There is a DTL-formula φ_0 such that φ_0 is valid on every dynamic preorder model $\mathfrak{M}$ where the underlying preorder is a disjoint union of finite preorders, but φ_0 is not valid on arbitrary dynamic preorder models.*

Proof. We give a formula ψ_0 which is satisfiable on a DPM but not in any point which generates a finite upset, then take $\varphi_0 := \neg\psi_0$.

Let us write $\Box\varphi$ for $\neg\Diamond\neg\varphi$ and $\blacksquare\varphi$ for $\neg\blacklozenge\neg\varphi$. Define $\psi_0 := \psi_1 \wedge \cdots \wedge \psi_4$ where

$$\psi_1 := \blacksquare\Box((p \to \bullet p) \wedge (\neg p \to \bullet\neg p) \wedge (q \to \bullet q))$$
$$\psi_2 := \Box((p \to \Diamond\neg p) \wedge (\neg p \to \Diamond p))$$
$$\psi_3 := \Box(p \to \blacklozenge(q \wedge \neg\Diamond(\neg p \wedge q)))$$
$$\psi_4 := \Box(\neg p \to \blacklozenge(q \wedge \neg\Diamond(p \wedge q))).$$

To see that ψ_0 is satisfied in a DPM, consider a model $\mathfrak{M}$ where $|\mathfrak{M}|$ consists of distinct points x_j^i for $i, j \in \mathbb{N}$. Define the preorder by $x_j^i \sqsubseteq_{\mathfrak{M}} x_{j'}^{i'}$ iff $i = i'$ and $j \leq j'$, and $f_{\mathfrak{M}}(x_j^i) = x_j^{i+1}$. Finally, define

$$V(p) := \{x_j^i \mid i \text{ is even}\} \quad \text{and} \quad V(q) := \{x_j^i \mid j \leq i\}.$$

This model is depicted in Figure 2. It is easy to check that $\mathfrak{M}$ satisfies ψ_0 in x_0^0.

Now suppose towards a contradiction that ψ_0 is satisfied in a point w of a DPM $\mathfrak{M}$, where the upset generated by $\{w\}$ is finite. By ψ_2, there are x_0 and y_0, successors of each other, in this upset satisfying p and $\neg p$ respectively. Write $x_i := f_{\mathfrak{M}}^i(x_0)$ and similar for y. By ψ_3 and ψ_4 there are m and n such that both x_m and y_n satisfy q, and do not satisfy $\Diamond(\neg p \wedge q)$ and $\Diamond(p \wedge q)$ respectively. If $m \leq n$, then, using ψ_1 we get that x_n satisfies p and q. By monotonicity of $f_{\mathfrak{M}}$, x_n is a successor of y_n, but then y_n satisfies $\Diamond(p \wedge q)$, contradiction. ◀

The failure of the locally finite model property is unfortunate, since any model-search procedure would need finite representations of models to work with. To circumvent the failure, we turn to an alternative, non-deterministic, semantics in the following section.

4 A Non-Deterministic Semantics

The relational semantics for DTL reviewed in Section 2 has two properties undesirable for our purposes. First, as we saw in the previous section, the relational semantics does not have the locally finite model property. Second, the information regarding satisfaction of formulas is *non-local*: determining whether a world satisfies a certain formula requires potentially arbitrary "lookaround". To overcome these limitations, we develop an alternative *non-deterministic* and *local* relational semantics for DTL in this section. This semantics is similar to and builds upon the *non-deterministic quasi-model semantics* for DTL by Fernández-Duque [14], which is itself a variation on the (deterministic) quasi-models used in [26]. Quasi-models had also been used to study many other modal logics prior to DTL, see e.g. [4, Section 6.4].

First note that we only care about satisfaction of a single formula or a finite set of formulas at a given time. The satisfaction of these only depends on that of their subformulas, which is still a finite set. Hence, we fix a finite subformula-closed set of formulas Σ. For convenience, we assume that if $\blacklozenge\varphi \in \Sigma$ then $\bullet\varphi \in \Sigma$.

Satisfaction information can now be "stored" locally at a point as the (finite) set of formulas from Σ satisfied at this point. If this set reflects the actual satisfaction of formulas in the point, in some model, then it obeys certain laws. The obvious syntactic laws are captured by the following definition.

▶ **Definition 4.1** (Σ-Type). *A subset $\Gamma \subseteq \Sigma$ is called a* Σ-type *iff*
- *if $\varphi_1 \wedge \varphi_2 \in \Sigma$ then $\varphi_1 \wedge \varphi_2 \in \Gamma$ iff $\{\varphi_1, \varphi_2\} \subseteq \Gamma$,*
- *if $\neg\varphi \in \Sigma$ then $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$, and*
- *if $\blacklozenge\varphi \in \Sigma$ and $\varphi \in \Gamma$, then $\blacklozenge\varphi \in \Gamma$.*

We write Ty_Σ *for the set of Σ-types.*

When we label the points in a preorder with these types, the presence of $\Diamond$-formulas in the type labels need not correspond with the usual interpretation of $\Diamond$ for the preorder. For our semantics, we require compatibility here, and call such labelled preorder a *quasi-state*. For technical reasons, we also add a set of designated points D to these structures, for which we later require temporal determinism.

▶ **Definition 4.2** (Quasi-state). *A* Σ-quasi-state $\mathfrak{S}$ *is a tuple $\langle|\mathfrak{S}|, \sqsubseteq_\mathfrak{S}, t_\mathfrak{S}, D_\mathfrak{S}\rangle$ where $\sqsubseteq_\mathfrak{S}$ is a preorder on $|\mathfrak{S}|$, $t_\mathfrak{S} \colon |\mathfrak{S}| \to \mathrm{Ty}_\Sigma$ a function called the* type labelling*, and $D_\mathfrak{S} \subseteq |\mathfrak{S}|$ a set of* deterministic points*, such that $D_\mathfrak{S}$ is downward closed and generates $\mathfrak{S}$, and for all $x \in |\mathfrak{S}|$ and $\Diamond\varphi \in \Sigma$,*

$$\Diamond\varphi \in t(x) \quad \textit{iff} \quad \exists y.\ x \sqsubseteq_\mathfrak{S} y, \varphi \in t(y).$$

A quasi-state is called deterministic *iff $D_\mathfrak{S} = |\mathfrak{S}|$.*

Since we have a single fixed set Σ in this and the following sections, we will mostly omit it from the notations and terminology, so that e.g. a *quasi-state* is a Σ-quasi-state. When clear from context, the subscripts on $\sqsubseteq$, t and D are also omitted.

For the non-deterministic semantics, the monotone function from the relational semantics is replaced with a continuous relation, where every point has at least one successor. Similar to the preorder, we impose a compatibility condition between the relation and the type labelling. We need two additional conditions to ensure designated points force temporal determinism as intended.

▶ **Definition 4.3** (Label-monotonicity). *Let $\langle L, \sqsubseteq\rangle$ be a preorder, and $\mathfrak{S} = \langle X, \sqsubseteq_X, l_X\rangle$ and $\mathfrak{T} = \langle Y, \sqsubseteq_Y, l_Y\rangle$ be L-labelled preorders. A relation $R \subseteq X \times Y$ is called* label-monotone *from $\mathfrak{S}$ to $\mathfrak{T}$ iff*

$$\forall\langle x, y\rangle \in R.\ l_X(x) \sqsubseteq l_Y(y).$$

If $\sqsubseteq$ is equality, we call R label-preserving *instead.*

For the purpose of label-monotonicity and label-preservation, we view a quasi-state $\mathfrak{S}$ as a labelled preorder $\langle|\mathfrak{S}|, \sqsubseteq_{\mathfrak{S}}, l_{\mathfrak{S}}\rangle$ by defining

$$l_{\mathfrak{S}} \colon |\mathfrak{S}| \to \mathrm{Ty}_{\Sigma} \times \mathbf{2} : x \mapsto \langle t_{\mathfrak{S}}(x), d_{\mathfrak{S}}(x)\rangle,$$

where $\mathbf{2} \coloneqq \{0, 1\}$ and $d_{\mathfrak{S}} \colon X \to \mathbf{2}$ is the characteristic function of $D_{\mathfrak{S}}$, i.e. $d_{\mathfrak{S}}(x) = 1$ if $x \in D_{\mathfrak{S}}$ and $d_{\mathfrak{S}}(x) = 0$ otherwise. For label-monotonicity, $\mathrm{Ty}_{\Sigma} \times \mathbf{2}$ is equipped with the order $\leq_{\mathrm{p}}$ defined by

$$\langle\Gamma, i\rangle \leq_{\mathrm{p}} \langle\Delta, j\rangle \quad \textit{iff} \quad (\Gamma = \Delta \textit{ and } i \leq_{\mathbf{2}} j),$$

where $\leq_{\mathbf{2}}$ is the standard order $0 \leq_{\mathbf{2}} 1$ on $\mathbf{2}$.

Furthermore, we call a relation between quasi-states $\mathfrak{S}$ and $\mathfrak{T}$ determinism-monotone *iff it is label-monotone between the $\mathbf{2}$-labelled preorders $\langle|\mathfrak{S}|, \sqsubseteq_{\mathfrak{S}}, d_{\mathfrak{S}}\rangle$ and $\langle|\mathfrak{T}|, \sqsubseteq_{\mathfrak{T}}, d_{\mathfrak{T}}\rangle$, where $\mathbf{2}$ is ordered by $\leq_{\mathbf{2}}$.* ⌟

▶ **Definition 4.4** (Temporal relation)**.** *Let $\mathfrak{S}$ and $\mathfrak{T}$ be Σ-quasi-states and let $F \subseteq |\mathfrak{S}| \times |\mathfrak{T}|$. Then F is called a* temporal relation from $\mathfrak{S}$ to $\mathfrak{T}$ *iff*

(i) $\mathrm{dom}(F) = |\mathfrak{S}|$, *where* $\mathrm{dom}(F) \coloneqq \{x \mid \exists y.\ \langle x, y\rangle \in F\}$,
(ii) *F is a continuous relation on the underlying preorders,*
(iii) *for all $\langle x, y\rangle \in F$ and $\blacklozenge\psi \in \Sigma$ such that $\psi \notin t_{\mathfrak{S}}(x)$, we have $\blacklozenge\psi \in t_{\mathfrak{S}}(x)$ iff $\blacklozenge\psi \in t_{\mathfrak{T}}(y)$,*
(iv) *for all $\langle x, y\rangle \in F$ and $\bullet\psi \in \Sigma$, we have $\bullet\psi \in t_{\mathfrak{S}}(x)$ iff $\psi \in t_{\mathfrak{T}}(y)$,*
(v) *F is determinism-monotone, and*
(vi) *for all $x \in D_{\mathfrak{S}}$, the set $F(x) \coloneqq \{y \mid \langle x, y\rangle \in F\}$ of F-images of x is a singleton.*

If $\mathfrak{S} = \mathfrak{T}$ then F is called a temporal relation on $\mathfrak{S}$*.*

Now we could define a model in the non-deterministic semantics to be a pair $\langle\mathfrak{S}, F\rangle$ where $\mathfrak{S}$ is a quasi-state and F is a temporal relation on $\mathfrak{S}$. However, it is useful to separate the "different points in time", so we have disjoint quasi-states, and the temporal relation always mapping one quasi-state to the next. We also consider "finite initial segments" of these structures, where only a finite number of "points in time" exist, after which there are no temporal successors any more.

▶ **Definition 4.5** (Quasi-sequence)**.** *A* quasi-sequence *is a tuple $\mathcal{S} = \langle I, \langle\mathfrak{S}_i \mid i \in I\rangle, \langle F_i \mid i + 1 \in I\rangle\rangle$ such that $I \subseteq \mathbb{N}$ is downward closed, i.e. $I = [0, n)$ for some $n \in \mathbb{N}$ or $I = \mathbb{N}$, $\mathfrak{S}_i$ is a quasi-state for each $i \in I$ and F_i is a temporal relation from $\mathfrak{S}_i$ to $\mathfrak{S}_{i+1}$ for each $i + 1 \in I$. The elements of I are called* indices, *and I is usually omitted from the notation. By convention, we write $\mathfrak{S}$ for the disjoint union of the $\mathfrak{S}_i$ and F or $F_{\mathcal{S}}$ for the disjoint union of the F_i. Moreover, we write $|\mathcal{S}|_i$ for $|\mathfrak{S}_i|$, and $I_{\mathcal{S}}$, $|\mathcal{S}|$, $\sqsubseteq_{\mathcal{S}}$, $t_{\mathcal{S}}$ and $D_{\mathcal{S}}$ for I, $|\mathfrak{S}|$, $\sqsubseteq_{\mathfrak{S}}$, $t_{\mathfrak{S}}$ and $D_{\mathfrak{S}}$ respectively. The quasi-sequence $\mathcal{S}$ is called*

short *iff I is finite, say $[0, n)$, in which case n is called the* length *of $\mathcal{S}$,*
long *iff $I = \mathbb{N}$,*
locally finite *iff $\mathfrak{S}_i$ is finite for each $i \in I$,*
locally s-small *for some function $s \colon \mathbb{N} \to \mathbb{N}$, iff $\#\mathfrak{S}_i \leq s(i)$ for each $i \in I$, where $\#\mathfrak{S}_i$ denotes the cardinality of $|\mathfrak{S}_i|$, and*
deterministic *iff $\mathfrak{S}_i$ is deterministic for each $i \in I$.*

Short quasi-sequences are thought of as "unfinished" structures that can still be extended to long quasi-sequences. A long quasi-sequence can be seen as the limit of (compatible) short quasi-sequences of increasing lengths, and we will construct long quasi-sequences this way later on.

A quasi-sequence $\mathcal{S}$ is said to satisfy a formula φ iff there exists $x \in |\mathcal{S}|$ such that $\varphi \in t_{\mathcal{S}}(x)$. If $\mathcal{S}$ has length at least $n \geq 1$, then the *initial segment of $\mathcal{S}$ of length n* is the restriction of $\mathcal{S}$ to indices $\{0, \dots, n-1\}$. For $n \in I_{\mathcal{S}}$, the *tail of $\mathcal{S}$ starting at index n* is the restriction of $\mathcal{S}$ to indices $\{m \in I_{\mathcal{S}} \mid m \geq n\}$ (then shifted so the indices start at 0 again).

A long deterministic quasi-sequence $\mathfrak{S}_-$ with temporal functions f_- can be seen as a DPM $\langle |\mathfrak{S}|, \sqsubseteq_{\mathfrak{S}}, f, V\rangle$ by defining $V(p) \coloneqq \{x \in X \mid p \in t_{\mathfrak{S}}(x)\}$. Then a formula $\varphi \in \Sigma$ free of $\blacklozenge$ is satisfied in a point $x \in |\mathfrak{S}|$ according to the relational DPM semantics iff it is satisfied in x according to the quasi-sequence semantics, i.e. $\varphi \in t_{\mathfrak{S}}(x)$. However, to extend this to formulas containing $\blacklozenge$, we need to guarantee the existence of eventual witnesses for $\blacklozenge$ formulas. We say that such witness *realizes* the $\blacklozenge$ *eventuality.*

▶ **Definition 4.6** (Eventuality, realization). *Let $\langle I, \mathfrak{S}_-, F_-\rangle$ be a quasi-sequence and $i \in I$. An* eventuality at stage i *is a pair $\langle x, \varphi\rangle$ such that $x \in |\mathfrak{S}_i|$ and φ a formula such that $\blacklozenge\varphi \in t_{\mathfrak{S}}(x)$. It is said to be* realized by y *for some $y \in |\mathfrak{S}|$ iff $\varphi \in t_{\mathfrak{S}}(y)$ and $y \in F^*(x)$, where $F^* \coloneqq \bigcup_{n\in\mathbb{N}} F^n$ denotes the reflexive transitive closure of F. In this case, it is also just called* realized*, or* realized at stage k *where k is such that $y \in |\mathfrak{S}_k|$.*

▶ **Definition 4.7** (Realizing quasi-sequence). *A long quasi-sequence is called* realizing *iff every eventuality in it is realized. We write* realizing sequence *for* long realizing quasi-sequence.

Now it is easy to see that the standard relational semantics is equivalent to the semantics of deterministic realizing sequences, in the sense that models from either semantics induce models in the other satisfying the same formulas in Σ.

▶ **Lemma 4.8.** *Let $\langle X, \sqsubseteq, f, V\rangle$ be a DPM. Define $t\colon X \to \mathrm{Ty}_\Sigma$ by $t(x) \coloneqq \{\varphi \in \Sigma \mid x \in [\![\varphi]\!]\}$. Then $\mathfrak{S} \coloneqq \langle X, \sqsubseteq, t, X\rangle$ is a quasi-state, f is temporal on $\mathfrak{S}$, and the long quasi-sequence consisting of $\mathfrak{S}_j \coloneqq \mathfrak{S}$ and $F_j \coloneqq f$ for each j is a deterministic realizing sequence.*

▶ **Lemma 4.9.** *Let $\mathfrak{S}_-$ with functions f_- be a deterministic realizing sequence. Then there exists a valuation V such that $t_{\mathfrak{S}}(x) = \{\varphi \in \Sigma \mid x \in [\![\varphi]\!]\}$, where the interpretation is taken in the DPM $\langle |\mathfrak{S}|, \sqsubseteq_{\mathfrak{S}}, f, V\rangle$.*

For completeness, it suffices to consider quasi-sequences where each quasi-state is rooted and "roots map to roots".

▶ **Definition 4.10** (Rooted quasi-sequence). *A preorder or quasi-state $\mathfrak{S}$ is called* rooted *iff there exists $r \in |\mathfrak{S}|$ which generates $\mathfrak{S}$. Such r is called a* root *of $\mathfrak{S}$. A quasi-sequence $\mathcal{S} = \langle I, \mathfrak{S}_-, F_-\rangle$ is called* rooted *iff each $\mathfrak{S}_i$ is rooted and for every $i+1 \in I$, there exist roots r_i of $\mathfrak{S}_i$ and r_{i+1} of $\mathfrak{S}_{i+1}$ such that $\langle r_i, r_{i+1}\rangle \in F_i$.*

Note that roots need not be unique, since every point in the same cluster as a root is also a root. However, since $D_{\mathfrak{S}}$ is downward closed and generates $\mathfrak{S}$, every root is deterministic, so r_{i+1} above is the unique temporal successor of r_i.

▶ **Lemma 4.11.** *Let $\varphi \in \Sigma$. Then φ is satisfied in a DPM iff it is satisfied in (the initial state of) a rooted deterministic realizing sequence.*

Proof sketch. By Lemmata 4.8 and 4.9, it suffices to show that if φ is satisfied in a deterministic realizing sequence $\mathcal{S} = \langle \mathfrak{S}_-, f_-\rangle$ then it is satisfied in a rooted one. By taking a tail of $\mathcal{S}$ we can assume, w.l.o.g. that $\varphi \in t_{\mathcal{S}}(x)$ for some $x \in |\mathcal{S}|_0$. Define a quasi-sequence $\mathcal{T}$ by restricting each quasi-state $\mathfrak{S}_i$ to the upset generated by $f^i(x)$. By monotonicity, the temporal functions restrict accordingly. ◀

5 The Limit Model Construction

In the previous section we saw, with Lemmata 4.8 and 4.9, that the standard relational semantics is equivalent to the semantics of *deterministic* realizing sequences. However, to obtain a locally finite model property, we need to relax this determinism condition. We achieve this by constructing deterministic quasi-sequences from certain non-deterministic ones. We proceed in two steps. First, we give a generic construction, a variant of the *limit model construction* introduced by Fernández-Duque [14]. Second, we identify a condition, *determinism-propagation*, which ensures that quasi-sequences satisfy the requirements for the generic construction to apply.

▶ **Definition 5.1** (Long path, Realizing path). *Let $\mathfrak{S}_{-}$ with relations F_{-} be a long quasi-sequence. A* long path *in it is a sequence $p\colon \mathbb{N} \to |\mathfrak{S}|$ such that $\langle p(i), p(i+1)\rangle \in F$ for all $i \in \mathbb{N}$. It is called* realizing *or a* realizing path *iff for every $i \in \mathbb{N}$, every eventuality of the form $\langle p(i), \varphi\rangle$ is realized by $p(j)$ for some $j \geq i$. Given a long path p, define its* shift $\sigma(p)$ *by $\sigma(p)(i) := p(i+1)$.*

▶ **Lemma 5.2.** *Let $\varphi \in \Sigma$, $\mathcal{S}$ be a long quasi-sequence, and P be a set of realizing paths in $\mathcal{S}$, such that:*
- **(i)** *P is closed under shifts, and*
- **(ii)** *for all $p \in P$ and $x \in |\mathcal{S}|$ with $p(0) \sqsubseteq_{\mathcal{S}} x$, there exists $p' \in P$ such that $p'(0) = x$ and $p(i) \sqsubseteq_{\mathcal{S}} p'(i)$ for all $i \in \mathbb{N}$.*

Then there exists a DPM $\mathfrak{M}$, such that for every $p \in P$ and $\varphi \in t_{\mathcal{S}}(p(0))$, this φ is satisfied in $\mathfrak{M}$ as well.

We call a set of long paths satisfying conditions (i) and (ii) from the lemma a *path system*.

Proof sketch. By Lemma 4.9 it suffices to construct a deterministic realizing sequence $\mathcal{T}$ satisfying φ. As the points in $\mathcal{T}$ we take the paths $p \in P$, similar to a tree unravelling in modal logic. We think of such a path p as its initial point $p(0)$, however, the rest of the path already *choses* which temporal successors to pick "in the future" if $\mathcal{S}$ is non-deterministic.

Formally, define $|\mathcal{T}|_i := \{p \in P \mid p(0) \in |\mathcal{S}|_i\}$, the set of paths in P starting at stage i, and note that then $|\mathcal{T}| = P$. Define a preorder $\sqsubseteq_{\mathcal{T}}$ on P by $p \sqsubseteq_{\mathcal{T}} p'$ iff $p(j) \sqsubseteq_{\mathcal{S}} p'(j)$ for all $j \in \mathbb{N}$, and notice that this relation does not relate points across different stages $|\mathcal{T}|_i$ and $|\mathcal{T}|_j$ with $i \neq j$. Next, define

$$t_{\mathcal{T}}\colon P \to \mathrm{Ty}_{\Sigma} : p \mapsto (t_{\mathcal{S}} \circ p)(0).$$

Because we want $\mathcal{T}$ to be deterministic, we need to set $D_{\mathcal{T}} := P$. Finally, equip $\mathcal{T}$ with the shift operator σ as the temporal function, which is well-defined by i.

Now ii is exactly what is necessary to make $\langle |\mathcal{T}|, \sqsubseteq_{\mathcal{T}}, t_{\mathcal{T}}, D_{\mathcal{T}}\rangle$ a quasi-state. It is easy to check that σ is a temporal function on it. The fact that the resulting quasi-sequence $\mathcal{T}$ is realizing follows from the fact that each element of P is realizing. ◀

Next, we identify a condition under which we can apply the previous construction. We instantiate P with the set of long paths which eventually hit the set of deterministic points $D_{\mathcal{S}}$. These paths are called *eventually deterministic*.

▶ **Definition 5.3** (Eventually deterministic). *Let $\mathcal{S}$ be a long quasi-sequence. A long path p in it is called* eventually deterministic *iff there exists $i \in \mathbb{N}$ such that $p(i) \in D_{\mathcal{S}}$.*

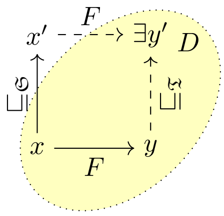

Figure 3 Diagrammatic representation of determinism-propagation, as defined in Definition 5.5. The yellow dotted ellipse represents the set of deterministic points D.

By determinism-monotonicity of temporal relations, we get $p(j) \in D_{\mathcal{S}}$ from i onward, for i as in the definition.

To instantiate P with these eventually deterministic paths, we need to check that they are realizing and form a path system. The former holds if the quasi-sequence $\mathcal{S}$ is realizing.

▶ **Lemma 5.4.** *Every eventually deterministic long path in a realizing sequence $\mathcal{S}$ is realizing.*

For the eventually deterministic paths to form a path system, we need an extra requirement on our quasi-sequences: *determinism-propagation*. Determinism-propagation is a relaxation of determinism, where we do not require *every* point to be deterministic, but we still want to be able to reach "enough" deterministic points as temporal successors. More concretely, if x' is a point above some deterministic point x, then we want x' to reach a deterministic point y' in a single temporal step, and moreover this y' should complete the continuity diagram (as in Figure 1) of x, x' and the unique temporal successor y of x, see Figure 3.

▶ **Definition 5.5** (Determinism-propagation). *Let $\mathfrak{S}$ and $\mathfrak{T}$ be quasi-states and F a temporal relation from $\mathfrak{S}$ to $\mathfrak{T}$. Then F is called* determinism-propagating *iff for every $x \in D_{\mathfrak{S}}$, $x' \in |\mathfrak{S}|$ and $y \in |\mathfrak{T}|$ with $x \sqsubseteq_{\mathfrak{S}} x'$ and $\langle x, y\rangle \in F$, there exists $y' \in D_{\mathfrak{T}}$ such that $y \sqsubseteq_{\mathfrak{S}} y'$ and $\langle x', y'\rangle \in F$. This condition is diagrammatically depicted in Figure 3. A quasi-sequence is called* determinism-propagating *iff every temporal relation in it is determinism-propagating.*

Note that when we remove the determinism requirements in this definition, we obtain the definition of continuity for F.

The relevance of determinism-propagation is that it makes the set of eventually deterministic paths satisfy condition ii of Lemma 5.2. Given a long path p and x above $p(0)$, it is easy, using continuity, to construct a path p' pointwise above p starting at x. The issue is to make an eventually deterministic one, given an eventually deterministic p. So assume $p(n)$ is a deterministic point. We construct p' up to and including $p'(n)$ as before. Now determinism-propagation gives us a temporal successor y' of $p'(n)$ that is above $p(n+1)$, like continuity would, but additionally guarantees that this y' can be chosen to be a deterministic point. From that point onward, p' is completely determined as y' and its temporal successors always have a unique temporal successor, and we obtain the required path p'. This leads to the following lemma.

▶ **Lemma 5.6.** *Let $\mathcal{S}$ be a long determinism-propagating quasi-sequence, and let P be the set of eventually deterministic paths in it. Then P forms a path system.*

Since every point is the successor of a deterministic one, it occurs in an eventually deterministic path. Hence, the DPM constructed by Lemma 5.2 satisfies all $\varphi \in \Sigma$ satisfied in $\mathcal{S}$. Thus, these quasi-sequences provide an equivalent semantics for DTL, and we call them *quasi-sequence models*.

▶ **Definition 5.7.** *A* quasi-sequence model *is a determinism-propagating realizing sequence.*

▶ **Proposition 5.8.** *A formula $\varphi \in \Sigma$ is satisfied in a quasi-sequence model iff it is satisfied in a dynamic preorder model.*

6 Tree Unravelling

One major shortcoming of the quasi-sequence semantics remains. When we introduced this semantics, we wanted it to be *local*. However, the requirement that quasi-sequence models be realizing is a non-local property, since we have no way of knowing how long it might take for an eventuality to be realized. As such, there is no way of checking whether a short quasi-sequence can be extended to a long realizing one.

In the next section, we provide a local condition which implies, for certain locally finite quasi-sequences, realization. The proof relies on trees, and in particular Kruskal's Tree Theorem. In this section, we prepare for this by considering a method for transforming ordered structures into tree-like ones: tree unravellings. Tree unravelling will again be used to establish a locally finite model property later on.

Tree unravelling is a technique from modal logic where a Kripke model $\mathfrak{M}$, e.g. a preorder model, is turned into a tree-like one $\mathfrak{M}'$, while preserving the satisfaction of formulas, see e.g. [4, 5]. This unravelling $\mathfrak{F}'$ is built by making many copies of points x in $\mathfrak{F}$ that capture the different paths through $\mathfrak{F}$ for getting to x, similar to the limit model construction from the previous section where we made copies of a point x for the different long paths going out of x. This means that there is a surjective *quotient map* q from the unravelling $\mathfrak{M}'$ to the original $\mathfrak{M}$ which preserves and reflects satisfaction of formulas.[5]

We make this precise in the setting of quasi-states and quasi-sequences. Let us first recall the definition of *tree-like* preorders.

▶ **Definition 6.1** (Tree-like preorder). *A preorder or quasi-state $\mathfrak{S}$ is called* tree-like *iff it is rooted and for every $x \in |\mathfrak{S}|$, the downset $\Diamond\{x\}$ generated by x*

- *is linear, i.e. for all $y, z \in \Diamond\{x\}$, either $y \sqsubseteq_{\mathfrak{S}} z$ or $z \sqsubseteq_{\mathfrak{S}} y$, and*
- *consists of finitely many clusters.*

A quasi-sequence is called tree-based *iff it is rooted and each of its quasi-states is tree-like.*

Note that the labelling that determines satisfaction is already internal in quasi-states, so the requirement on the quotient map q becomes just label-preservation.[6] However, for quasi-sequences we want the analogue of q to additionally preserve and reflect the realization of eventualities, so that the unravelling of a realizing sequence is again realizing. This is achieved by requiring q to commute with the temporal relations of the quasi-sequences it maps between, leading to the following definition.

▶ **Definition 6.2** (Quasi-sequence morphism). *Let $\mathcal{S} = \langle I, \mathfrak{S}_-, F_- \rangle$ and $\mathcal{T} = \langle I', \mathfrak{T}_-, G_- \rangle$ be quasi-sequences with $I = I'$. A* quasi-sequence morphism *between $\mathcal{S}$ and $\mathcal{T}$ is a sequence of functions $f_i \colon |\mathfrak{S}_i| \to |\mathfrak{T}_i|$ for $i \in I$ such that $G_i \circ f_i = f_{i+1} \circ F_i$ for $i + 1 \in I$, i.e. the following diagram commutes:*

[5] Usually this is achieved by proving that this q is a *p-morphism*, see Definition 8.1.

[6] Technically, preservation of the type labelling suffices, but by convention we consider "full" label-preservation.

$$\begin{tikzcd}
\mathfrak{S}_0 \arrow[r, "F_0"] \arrow[d, "f_0"] & \mathfrak{S}_1 \arrow[r, "F_1"] \arrow[d, "f_1"] & \mathfrak{S}_2 \arrow[r] \arrow[d, "f_2"] & \cdots \\
\mathfrak{T}_0 \arrow[r, "G_0"'] & \mathfrak{T}_1 \arrow[r, "G_1"'] & \mathfrak{T}_2 \arrow[r] & \cdots
\end{tikzcd}$$

It is called surjective, monotone, label-monotone*, an* isomorphism*, etc. iff each of the f_i is so. By convention, we also write f for the disjoint union of the f_i.*

We describe a tree unravelling which preserves local finiteness, i.e. if $\mathcal{S}$ is locally finite then so is its unravelling $\mathcal{T}$. To define the tree unravelling, we need to introduce the *skeleton* of a preorder: the quotient of the preorder under the induced equivalence relation $\equiv$.

▶ **Definition 6.3** (Skeleton)**.** *Let $\mathfrak{P}$ be a preorder. Write Y for the set of clusters of $\mathfrak{P}$. Note that for $A, B \in Y$, $x \in A$ and $y \in B$, whether $x \sqsubseteq_{\mathfrak{P}} y$ holds is independent of the particular choice of x and y. Therefore, $\sqsubseteq_{\mathfrak{P}}$ induces a relation $\leq$ on Y. Now $\langle Y, \leq\rangle$ is called the* skeleton *of $\mathfrak{P}$.*

Clearly, the skeleton of a preorder is a partial order. The operation of taking skeletons is also functorial, in the sense that a monotone function between preorders induces a monotone function between their skeletons.

▶ **Construction 6.4** (Unravelling quasi-states)**.** Let $\mathfrak{S}$ be a rooted quasi-state, and $\mathfrak{P}$ its skeleton. Define its tree unravelling $\mathfrak{T}$ as follows. Let P_n, for $n > 0$, denote the set of sequences p of length n of points in $\mathfrak{P}$ such that $p(0)$ is a root of $\mathfrak{P}$ and for every i with $i + 1 < n$, the point $p(i + 1)$ is a strict successor of $p(i)$. Define

$$|\mathfrak{T}| \coloneqq \{\langle p, x\rangle \mid n \in \mathbb{N}, p \in P_{n+1}, x \in p(n)\}.$$

Define a preorder $\sqsubseteq_{\mathfrak{T}}$ on it by $\langle p, x\rangle \sqsubseteq_{\mathfrak{T}} \langle p', x'\rangle$ iff p is an initial segment of p'. This way $\mathfrak{T}$ is a tree-like preorder, and it is finite if $\mathfrak{S}$ is so.

Define the quotient map $q\colon |\mathfrak{T}| \to |\mathfrak{S}|$ by $q(\langle p, x\rangle) \coloneqq x$. Since q is surjective, $\mathfrak{S}$ induces a unique labelling on $\mathfrak{T}$ that makes q label-preserving. It is easy to check that $\mathfrak{T}$ is a quasi-state.

To unravel a rooted quasi-sequence, we apply the previous quasi-state unravelling element-wise to its quasi-states. It remains to "lift" the temporal relations. The approach for this differs between deterministic and non-deterministic points.

▶ **Construction 6.5** (Unravelling quasi-sequences)**.** Given a rooted quasi-sequence $\mathcal{S} = \langle I, \mathfrak{S}_-, F_-\rangle$, let $\mathfrak{T}_i$ be the tree unravelling of $\mathfrak{S}_i$, as in Construction 6.4, with quotient map q_i. Write f_i for the restriction of F_i to deterministic points, and $\tilde{f}_i$ for the induced function between skeletons. Define a temporal relation G_i from $\mathfrak{T}_i$ to $\mathfrak{T}_{i+1}$ as follows. Let $\langle p, x\rangle \in |\mathfrak{T}_i|$, where p has length say n, i.e. $p \in P_n$ from Construction 6.4. If $\langle p, x\rangle$ is deterministic, set $\langle\langle p, x\rangle, \langle p', x'\rangle\rangle \in G_i$ iff p' also has length n, $\tilde{f}_i(p(j)) = p(j)$ for all $j < n$, and $f_i(x) = x'$. Otherwise, set $\langle\langle p, x\rangle, \langle p', x'\rangle\rangle \in G_i$ iff $\langle x, x'\rangle \in F_i$. Define the tree unravelling of $\mathcal{S}$ to be $\langle I, \mathfrak{T}_-, G_-\rangle$, and note that by construction the sequence q_- is a quasi-sequence morphism.

We call a quasi-sequence *tree-unravelled* iff it is (isomorphic to) the tree unravelling of a rooted quasi-sequence. Let us note four properties of the tree unravelling.

▶ **Lemma 6.6.** *Let $\mathcal{S}$ be a rooted quasi-sequence, and $\mathcal{T}$ its tree unravelling, with quotient map q. Then*

(i) *q is a surjective monotone label-preserving quasi-sequence morphism from $\mathcal{T}$ to $\mathcal{S}$, and the converse relation q^{-1} is continuous,*

(ii) *$\mathcal{S}$ is locally finite, realizing or deterministic, respectively, iff $\mathcal{T}$ is so,*

(iii) *the temporal relation of $\mathcal{T}$ maps downward closed subsets of $D_{\mathcal{T}}$ to downward closed sets, and*

(iv) *for any $x \in |\mathcal{T}|$ and successor y of x, there exists an immediate successor y' of x with $q(y) = q(y')$.*

7 Progress

In this section, we provide a local condition for locally finite quasi-sequences which implies realization. For this, we apply Kruskal's Tree Theorem to the tree unravelling from the previous section.

Suppose for a moment that a long locally finite quasi-sequence $\mathcal{S}$ contains infinitely many eventualities, which we preorder by their stage. Since, by local finiteness, there are only finitely many eventualities per stage, being realizing becomes equivalent to *progressing* infinitely often: realizing a minimal realized eventuality not realized.

▶ **Definition 7.1** (Progress). *Let $\mathcal{S}$ be a quasi-sequence and $k \in I_{\mathcal{S}}$ an index in it. Let j be the greatest number $\leq k$ such that all eventualities at stages $< j$ are realized by stage k, where we note such a j exists, since for $j = 0$ the condition holds vacuously. Then the sequence is said to* progress *at stage k iff there exists an eventuality $\langle x, \varphi\rangle$ at stage j which is realized at stage k but not at any stage $< k$. In this case the progress is said to be* based on *the stage j eventuality $\langle x, \varphi\rangle$.*

The following lemma makes our observation precise, and also handles the case where there are only finitely many eventualities.

▶ **Lemma 7.2.** *Let $\mathcal{S}$ be a long locally finite quasi-sequence. Then the following are equivalent:*

(i) *$\mathcal{S}$ is realizing,*

(ii) *either there are infinitely many stages where $\mathcal{S}$ progresses or $\mathcal{S}$ has finitely many eventualities, and*

(iii) *there are infinitely many stages k such that either $\mathcal{S}$ progresses at stage k or the initial segment of $\mathcal{S}$ of length $k + 1$ is realizing.*

Therefore, to guarantee a quasi-sequence is realizing, we only need it to progress regularly. However, there is no obvious fixed "rate" at which we might hope to have progress, e.g. at least once every n stages. There is, however, a combinatoral criterion, based on certain continuous and label-monotone relations between quasi-states, which we call *quasi-state simulations.*

▶ **Definition 7.3** (Quasi-state simulation). *Let $\mathfrak{S}$ and $\mathfrak{T}$ be quasi-states and let $E \subseteq |\mathfrak{S}| \times |\mathfrak{T}|$. Then E is called a* simulation of $\mathfrak{S}$ in $\mathfrak{T}$ *iff*

- $\mathrm{dom}(E) = |\mathfrak{S}|$,
- *E is continuous,*
- *E is label-monotone, and*
- *for all $x \in |\mathfrak{S}|$, if $x \in D_{\mathfrak{S}}$ then $E(x)$ is singleton.*

These conditions ensure precisely that the composition of a simulation with a temporal relation is temporal.

▶ **Lemma 7.4.** *Let $\mathfrak{S}_i$ be quasi-states for $i \in \{1,2,3\}$.*

- *If E is a simulation of $\mathfrak{S}_1$ in $\mathfrak{S}_2$ and F is a temporal relation from $\mathfrak{S}_2$ to $\mathfrak{S}_3$, then $F \circ E$ is a temporal relation from $\mathfrak{S}_1$ to $\mathfrak{S}_3$.*
- *If F is a temporal relation from $\mathfrak{S}_1$ to $\mathfrak{S}_2$ and E is a simulation of $\mathfrak{S}_2$ in $\mathfrak{S}_3$, then $E \circ F$ is a temporal relation from $\mathfrak{S}_1$ to $\mathfrak{S}_3$.*

As a direct consequence, we get the following lemma, which allows to "collapse" the part of a quasi-sequence between stage i and j if quasi-state $i < j$ is simulated in quasi-state j.

▶ **Lemma 7.5.** *Let $\mathcal{S} = \langle \mathfrak{S}_-, F_- \rangle$ be a quasi-sequence, $i, j \in I_{\mathcal{S}}$ such that $i < j$ and E a simulation of $\mathfrak{S}_i$ in $\mathfrak{S}_j$. Then*

$$\mathfrak{S}_0 \xrightarrow{F_0} \mathfrak{S}_1 \xrightarrow{F_1} \cdots \xrightarrow{F_{i-1}} \mathfrak{S}_i \xrightarrow{F_j \circ E} \mathfrak{S}_{j+1} \xrightarrow{F_{j+1}} \cdots$$

is a quasi-sequence.

The resulting quasi-sequence is called the $\langle i, j, E\rangle$*-jump reduction* of $\mathcal{S}$. Given a realizing sequence with infinitely many eventualities, we can perform $\langle i, j, E\rangle$-jump reductions whenever, for each k with $i < k \leq j$, no progress occurs at this stage k. Then the resulting long quasi-sequence is still realizing. Additionally, it now has the property that whenever quasi-state i is simulated in quasi-state $j > i$, then progress occurs "in-between". We call such quasi-sequence *reduced*. In the following definition we also account for the case of finitely many eventualities.

▶ **Definition 7.6** (Reduced quasi-sequence)**.** *A quasi-sequence $\mathcal{S} = \langle \mathfrak{S}_-, F_- \rangle$ is called* reduced *iff for every $i, j \in I_{\mathcal{S}}$ such that $i < j$, if there exists a simulation of $\mathfrak{S}_i$ in $\mathfrak{S}_j$, then either*

- *there exists k with $i < k \leq j$ such that $\mathcal{S}$ progresses at stage k, or*
- *every eventuality at stage $\leq i$ is realized at a stage $\leq i$.*

Note that this property of being reduced is exactly a progress "rate" condition as we sought. This depends on the existence of "enough", i.e. infinitely many, pairs $i < j$ such that $\mathfrak{S}_i$ is simulated in $\mathfrak{S}_j$. This is where Kruskal's Tree Theorem comes into play.

Using Kruskal's Theorem we show that a long locally finite tree-based quasi-sequence has infinitely many of these simulation pairs. Thus, if the quasi-sequence is additionally reduced, then it is realizing.

Kruskal's Theorem is a statement about *well-quasi-orders*, preorders in which there are no infinite nowhere-ascending sequences.

▶ **Definition 7.7** (Well-quasi-order)**.** *A preorder $\langle X, \sqsubseteq \rangle$ is called a* well-quasi-order *iff for every infinite sequence $f \colon \mathbb{N} \to X$ there exist $i, j \in \mathbb{N}$ with $i < j$ and $f(i) \sqsubseteq f(j)$.*

Kruskal's original theorem [30] concerns an intransitive notion of trees, but a variant for tree-like preorders easily follows. A function ι between preorders or quasi-states $\mathfrak{S}$ and $\mathfrak{T}$ is called an *embedding* iff it is injective, monotone and *order-reflecting*, i.e. for $x, x' \in |\mathfrak{S}|$ with $\iota(x) \sqsubseteq_{\mathfrak{T}} \iota(x')$ then $x \sqsubseteq_{\mathfrak{S}} x'$.

▶ **Theorem 7.8** (Kruskal's Tree Theorem [30])**.** *Let $\langle L, \sqsubseteq \rangle$ be a well-quasi-order. For L-labelled preorders $\mathfrak{S} = \langle X, \sqsubseteq_X, l_X \rangle$ and $\mathfrak{T} = \langle Y, \sqsubseteq_Y, l_Y \rangle$, write $\mathfrak{S} \sqsubseteq_{\mathrm{emb}} \mathfrak{T}$ iff there exists a label-monotone embedding from $\mathfrak{S}$ to $\mathfrak{T}$. Then $\sqsubseteq_{\mathrm{emb}}$ forms a well-quasi-order on the class of finite L-labelled tree-like preorders.*

Recall that label-monotonicity for quasi-states was defined by viewing quasi-states as labelled preorders. Therefore, the theorem applies to quasi-states as well. Since L is finite, $\leq_{\mathrm{p}}$ is a well-quasi-order. Thus, by Kruskal's Theorem, finite tree-like quasi-states are well-quasi-ordered by label-monotone embeddibility. Repeatedly applying this result, we obtain the following.

▶ **Lemma 7.9.** *Every long locally finite tree-based reduced quasi-sequence is realizing.*

Proof sketch. Let $\mathcal{S}$ be a long reduced locally finite tree-based quasi-sequence, and write $\mathfrak{S}_{-}$ for the sequence of quasi-states. By Kruskal's Tree Theorem, applied to quasi-states, there exist i, j with $j > i$ and a label-monotone embedding ι from $\mathfrak{S}_i$ into $\mathfrak{S}_j$. Every label-monotone embedding is a quasi-state simulation, so $\mathfrak{S}_i$ is simulated in $\mathfrak{S}_j$. Since $\mathcal{S}$ is reduced, there is either progress at a stage k with $i < k \leq j$ or every eventuality at a stage $\leq i$ is realized at a stage $\leq i$. Applying Kruskal's Theorem repeatedly, for the sequence starting at $\mathfrak{S}_{j+1}$, we obtain Lemma 7.2 (iii). Thus, $\mathcal{S}$ is realizing. ◀

Using tree unravelling, we can relax the assumption that $\mathcal{S}$ is tree-based a bit.

▶ **Definition 7.10.** *A quasi-state $\mathfrak{S}$ is called* determinism-tree-like *iff the restriction of $\mathfrak{S}$ to $D_{\mathfrak{S}}$ is tree-like. A quasi-sequence is* determinism-tree-like *iff each quasi-state in it is.*

▶ **Proposition 7.11.** *Let $\mathcal{S}$ be a rooted long reduced locally finite determinism-tree-like quasi-sequence. Then $\mathcal{S}$ is realizing.*

Proof sketch. Let $\mathcal{T}$ be the tree unravelling of $\mathcal{S}$, with quotient map q_{-}. Then $\mathcal{T}$ is locally finite. It suffices to prove $\mathcal{T}$ is reduced, for if it is then by Lemma 7.9 $\mathcal{T}$ is realizing and hence by Lemma 6.6 (ii) so is $\mathcal{S}$.

Write $\mathfrak{S}_{-}$ for the quasi-state of $\mathcal{S}$ and $\mathfrak{T}_{-}$ for those of $\mathcal{T}$. To prove $\mathcal{T}$ is reduced, suppose E is a simulation of $\mathfrak{T}_i$ in $\mathfrak{T}_j$ for some $i < j$. Consider the restriction $\mathfrak{T}'_i$ of $\mathfrak{T}_i$ consisting of points $\langle p, x\rangle$ such that either

- $p(j+1)$ is an immediate successor of $p(j)$, or
- $p(j)$ has no strict successor that is a deterministic point,

for every $j < n-1$, where n denotes the length of p as a sequence. This $\mathfrak{T}'_i$ is still a tree-like quasi-state, and the restriction of q_i to it is still surjective onto $\mathfrak{S}_i$, but every deterministic point in $\mathfrak{S}_i$ has a unique q_i-preimage in $\mathfrak{T}'_i$.

It is straightforward to check that the restriction E' of E to $\mathfrak{T}'_i$ is a simulation of $\mathfrak{T}'_i$ in $\mathfrak{T}_j$. Now define

$$E'' := \{\langle q_i(x), q_i(y)\rangle \mid \langle x, y\rangle \in E'\}.$$

It follows from continuity and label-preservation of q_i^{-1} and the uniqueness of q_i-preimages for deterministic points that E'' is a simulation of $\mathfrak{S}_i$ in $\mathfrak{S}_j$.

Since $\mathcal{S}$ is reduced, this means that one of the two items in Definition 7.6 holds for $\mathcal{S}$, but either property lifts to $\mathcal{T}$. Therefore, $\mathcal{T}$ is reduced. ◀

8 Fine's Selection Method

The final ingredient for the enumeration procedure is a *locally finite model property*: given a formula $\varphi \in \Sigma$ satisfied in a potentially huge DPM, we want to construct a locally s-small quasi-sequence model satisfying φ, for some fixed computable bound function s. In modal logic, there are two main approaches to obtain finite model properties: filtration and selection. In this section, we recall Fine's [22, Section 6] selection method, which we will build upon in the next two sections to obtain the locally finite model property.

▶ **Definition 8.1** (Bisimulation). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be preorder models. A* bisimulation *between $\mathfrak{M}$ and $\mathfrak{N}$ is a relation $Z \subseteq |\mathfrak{M}| \times |\mathfrak{N}|$ such that both Z and the converse relation $Z^{\mathrm{op}} := \{\langle y, x\rangle \mid \langle x, y\rangle \in Z\}$ are continuous, and $x \in V_{\mathfrak{M}}(p) \iff y \in V_{\mathfrak{N}}$ for every atomic proposition p and $\langle x, y\rangle \in Z$. Points x and y are called* bisimilar, *denoted $\langle \mathfrak{M}, x\rangle \rightleftharpoons \langle \mathfrak{N}, y\rangle$, iff there exists a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{N}$ such that $\langle x, y\rangle \in Z$. A bisimulation that is a surjective function $|\mathfrak{M}| \to |\mathfrak{N}|$ is called a* p-morphism.

Bisimulation, bisimilarity and p-morphism definitions for quasi-states are similar, with the modification that preservation of atomic formulas is replaced with label-preservation.

Let us fix a finite subformula-closed set of unimodal formulas Σ (so free of $\bullet$ and $\blacklozenge$).

▶ **Definition 8.2** (Σ-sub-p-morphism). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be preorder models, and f a partial function from $\mathfrak{M}$ to $\mathfrak{N}$, and denote the restriction of $\mathfrak{M}$ to* $\mathrm{dom}(f)$ *by $\mathfrak{M}'$. Then f is called a Σ-sub-p-morphism, iff it is a p-morphism from $\mathfrak{M}'$ to $\mathfrak{N}$ and for every $x \in |\mathfrak{M}|$ there exists $y \in |\mathfrak{M}'|$ such that $x \sqsubseteq_{\mathfrak{M}} y$ and x and y satisfy the same Σ-formulas, i.e. for every $\varphi \in \Sigma$,*

$$x \in [\![\varphi]\!]_{\mathfrak{M}} \iff y \in [\![\varphi]\!]_{\mathfrak{M}}.$$

In this case $\mathfrak{N}$ is called a Σ-sub-p-morphic image *of $\mathfrak{M}$.*

A Σ-sub-p-morphism f preserves satisfaction of formulas in Σ, i.e. for every $\varphi \in \Sigma$ and $x \in \mathrm{dom}(f)$,

$$x \in [\![\varphi]\!]_{\mathfrak{M}} \iff f(x) \in [\![\varphi]\!]_{\mathfrak{N}}.$$

Hence, whenever φ is satisfied in a point of $\mathfrak{M}$ then it is satisfied in a point of $\mathfrak{N}$.

Fine's Selection Theorem now states that any preorder model $\mathfrak{M}$ has a finite Σ-sub-p-morphic image. In particular, every unimodal formula that is satisfied in preorder model is satisfied in a finite one. Besides the original paper [22, Theorem 6.6], a proof can be found in [5, Theorem 9.34].

▶ **Theorem 8.3** (Fine's Selection Theorem; [22, Theorem 6.6]). *Let Σ be a finite subformula-closed set of unimodal formulas and $\mathfrak{M}$ a preorder model. Then $\mathfrak{M}$ has a finite Σ-sub-p-morphic image, with size primitive recursive (hence computable) in that of Σ.*

To apply this theorem to quasi-states, we view a quasi-state as a preorder model. Let Σ be a finite subformula-closed set of DTL-formulas. Now consider unimodal formulas where we take as the set of atomic propositions Σ plus one extra atomic proposition d. For $\varphi \in \Sigma$ we write $(\!|\varphi|\!)$ for φ seen as an atomic proposition in this new language.

We can now view a Σ-quasi-state $\mathfrak{S}$ as a preorder model $\langle |\mathfrak{S}|, \sqsubseteq_{\mathfrak{S}}, V_{\mathfrak{S}}\rangle$ where we define

$$V_{\mathfrak{S}}((\!|\varphi|\!)) := \{x \in |\mathfrak{S}| \mid \varphi \in t_{\mathfrak{S}}(x)\}$$

for $\varphi \in \Sigma$, and $V_{\mathfrak{S}}(d) := D_{\mathfrak{S}}$. The condition on the type labelling for quasi-states is now equivalent to the condition that $[\![\Diamond(\!|\varphi|\!)]\!] = [\![(\!|\Diamond\varphi|\!)]\!]$ for all $\Diamond\varphi \in \Sigma$. Bisimulation of quasi-states now coincides with bisimulation on them viewed as preorder models.

9 Selection for Quasi-Sequences

In this section we give a construction, based on Fine's Selection Theorem, to obtain the locally finite model property. In particular, given a tree-unravelled deterministic realizing sequence, we construct a locally small quasi-sequence model.

▶ **Construction 9.1.** Let $\mathcal{S} = \langle \mathfrak{S}_-, F_- \rangle$ be a quasi-sequence of length at least 3 such that

- $|\mathfrak{S}_0|$ is finite,
- $F_0(D_0)$ is downward closed in $\mathfrak{S}_1$, and
- the tail of $\mathcal{S}$ starting at index 1 is deterministic and tree-unravelled.

Since $\mathfrak{S}_1$ is deterministic, we write f_1 for F_1. We construct a finite quasi-state $\tilde{\mathfrak{S}}_1$ and temporal relations $\tilde{F}_0$ from $\mathfrak{S}_0$ to $\tilde{\mathfrak{S}}_1$ and $\tilde{F}_1$ from $\tilde{\mathfrak{S}}_1$ to $\mathfrak{S}_2$, as depicted in the below diagram:

$$\begin{tikzcd}
\mathfrak{S}_0 \arrow[r, "F_0"] \arrow[dr, dashed, "\tilde{F}_0"'] & \mathfrak{S}_1 \arrow[r, "f_1"] & \mathfrak{S}_2 \\
& \tilde{\mathfrak{S}}_1 \arrow[ur, dashed, "\tilde{F}_1"'] &
\end{tikzcd}$$

Note that by Definition 4.4 (vi), $F_0(D_0)$ has size at most that of D_0, which is finite since $|\mathfrak{S}_0|$ is so. Also recall that $F_0(D_0)$ is downward closed. Let $\rightleftharpoons_1$ denote the largest bisimilation on $\mathfrak{S}_1$. Write $\sqsubseteq_i$ for the preorder relation of $\mathfrak{S}_i$.

Let $y_0 \in D_0$ and $x_0 \in |\mathfrak{S}_0| \setminus D_0$ such that $y_0 \sqsubseteq_0 x_0$. By continuity, there exists $x_1 \in F_0(x_0)$ such that $F_0(y_0) \sqsubseteq_1 x_1$. By Lemma 6.6 (iv), there exists an immediate successor y'' of $F_0(y_0)$ such that $y'' \rightleftharpoons_1 x_1$. For any y_0, x_0 as given, choose such y''_{y_0,x_0}, and write

$$Y''_1 \coloneqq \{y''_{y,x} \mid y \in D_0, x \in |\mathfrak{S}_0| \setminus D_0, y \sqsubseteq_0 x\}.$$

Define $Y_1 \coloneqq F_0(D_0) \cup Y''_1$. Since $F_0(D_0)$ is downward closed and every element $y'' \in Y''_1$ is an immediate successor of some $y \in F_0(D_0)$, we get that Y_1 is downward closed as well.

Define $F_0^{\Delta} \subseteq |\mathfrak{S}_0| \times |\mathfrak{S}_1|$ by $\langle x, y \rangle \in F_0^{\Delta}(x)$ iff

$$\begin{aligned}
&(x \in D_0 \text{ and } \langle x, y\rangle \in F_0(x)) \text{ or} \\
&\quad (x \notin D_0 \text{ and } \exists y' \in F_0(x).\ y \rightleftharpoons_1 y' \text{ and } \exists y'' \in Y''_1.\ y'' \sqsubseteq_1 y).
\end{aligned}$$

View $\mathfrak{S}_1$ as a preorder model as described in the previous section, and denote this model by $\mathfrak{M}$. We extend $\mathfrak{M}$ to a model $\mathfrak{M}^+$ with additional fresh atomic propositions $|x\rangle$ for every $x \in Y_1$ and every $x \in |\mathfrak{S}_0| \setminus D_0$ as follows. Write $\mathbb{P}^+$ for this new set of atomic propositions, that is,

$$\mathbb{P}^+ \coloneqq \{|\varphi\rangle \mid \varphi \in \Sigma\} \cup \{d\} \cup \{|x\rangle \mid x \in Y_1 \cup |\mathfrak{S}_0| \setminus D_0\}.$$

For $y \in Y_1$ and $x \in |\mathfrak{S}_0| \setminus D_0$, define

$$V_{\mathfrak{M}^+}(|y\rangle) \coloneqq \{y\} \quad \text{and} \quad V_{\mathfrak{M}^+}(|x\rangle) \coloneqq F_0^{\Delta}(x),$$

and $V_{\mathfrak{M}^+}(p) \coloneqq V_{\mathfrak{M}}(p)$ for atomic "old" atomic propositions p of $\mathfrak{M}$. Let Σ^+ be the set of unimodal formulas over atomic propositions $\mathbb{P}^+$ containing

(i) p for every $p \in \mathbb{P}^+$,
(ii) $|\varphi\rangle \wedge |\psi\rangle$ for every $\varphi \wedge \psi \in \Sigma$,
(iii) $\neg|\varphi\rangle$ for every $\neg\varphi \in \Sigma$, and
(iv) $\Diamond|\varphi\rangle$ for every $\Diamond\varphi \in \Sigma$.

Notice that Σ^+ is finite. By Fine's Selection Theorem, there exists a finite Σ^+-sub-p-morphic image $\mathfrak{N}$ of $\mathfrak{M}^+$, whose size is primitive recursive in that of Σ^+.

Let q_1 be a surjective Σ^+-sub-p-morphism from $\mathfrak{M}^+$ to $\mathfrak{N}$. Define $\tilde{D}_1 \coloneqq q_1(Y_1)$. Note that $Y_1 \subseteq \mathrm{dom}(q_1)$, and $q^{-1}(\tilde{y})$ is a singleton for $\tilde{y} \in \tilde{D}_1$.

$$\begin{tikzcd}
\mathfrak{S}_0 \arrow[r, "F_0"] & \mathfrak{S}_1 \arrow[r, "f_1"] & \mathfrak{S}_2 \arrow[dd, equal] \\
 & \operatorname{dom} q_1 \arrow[u, hook, "\subseteq"'] \arrow[d, two heads, "q_1"] & \\
\mathfrak{S}_0 \arrow[uur, "F_0^{\Delta}"] \arrow[r, "\tilde{F}_0"'] & \tilde{\mathfrak{S}}_1 \arrow[r, "\tilde{F}_1"'] & \mathfrak{S}_2
\end{tikzcd}$$

Figure 4 Commutative diagram for Construction 9.1.

Define

$$\tilde{t}_1(\tilde{x}) \coloneqq \{\varphi \in \Sigma \mid x \in [\![|\varphi\rangle]\!]_{\mathfrak{N}}\}$$

for $\tilde{x} \in |\mathfrak{N}|$. Since q_1 preserves unimodal formulas, using the elements of Σ^+ from Items (ii) and (iii) it follows that $\tilde{t}_1(\tilde{x})$ is a Σ-type. Define $\tilde{\mathfrak{S}}_1 \coloneqq \langle |\mathfrak{N}|, \sqsubseteq_{\mathfrak{N}}, \tilde{t}_1, \tilde{D}_1 \rangle$, $\tilde{F}_0 \coloneqq q_1 \circ F_0^{\Delta}$ and $\tilde{F}_1 \coloneqq f_1 \circ q_1^{-1}$. The defining commutative diagram for $\tilde{F}_0$ and $\tilde{F}_1$ is shown in Figure 4. ⌟

In the following lemma we prove that these are temporal relations, as claimed, and, if $\mathcal{S}$ had length at least 4 then the construction applies again to the quasi-sequence

$$\tilde{\mathfrak{S}}_1 \xrightarrow{\tilde{F}_1} \mathfrak{S}_2 \xrightarrow{f_2} \mathfrak{S}_3 \longrightarrow \cdots$$

where we write f_2 for F_2 since $\mathfrak{S}_2$ is deterministic. We conclude the lemma with two properties of $\tilde{\mathfrak{S}}_1$ that remind of the assumptions of Proposition 7.11.

▶ **Lemma 9.2.** *In Construction 9.1,*
- **(i)** $\tilde{\mathfrak{S}}_1$ *is a determinism-tree-like quasi-state,*
- **(ii)** $\tilde{\mathfrak{S}}_1$ *has size bounded by a primitive recursive (hence computable) function in the size of* Σ *and* $\mathfrak{S}_0$*,*
- **(iii)** *any* $\varphi \in \Sigma$ *is satisfied in a world of* $\mathfrak{S}_1$ *iff it is satisfied in a world of* $\tilde{\mathfrak{S}}_1$*,*
- **(iv)** $\tilde{F}_0$ *is a determinism-propagating temporal relation from* $\mathfrak{S}_0$ *to* $\tilde{\mathfrak{S}}_1$*,*
- **(v)** $\tilde{F}_1$ *is a temporal relation from* $\tilde{\mathfrak{S}}_1$ *to* $\mathfrak{S}_2$*, and*
- **(vi)** $\tilde{F}_1(\tilde{D}_1)$ *is a downward closed in* $\mathfrak{S}_2$*.*

If $\mathcal{S}$ is a realizing sequence, then applying Construction 9.1 preserves this realization.

▶ **Lemma 9.3.** *Let* $\mathcal{S} = \langle \mathfrak{S}_-, F_- \rangle$ *be a long quasi-sequence and* $n > 0$ *such that the tail* $\mathcal{S}$ *starting at index* $n-1$ *satisfies the assumptions of Construction 9.1. Let* $\tilde{\mathfrak{S}}_n$ *and relations* $\tilde{F}_{n-1}$ *and* $\tilde{F}_n$ *be the result of applying the construction. Denote by* $\mathcal{T}$ *the long quasi-sequence that equals* $\mathcal{S}$ *except for the indices* $\{n-1, n, n+1\}$ *where it is given by*

$$\tilde{\mathfrak{S}}_{n-1} \xrightarrow{\tilde{F}_{n-1}} \tilde{\mathfrak{S}}_n \xrightarrow{\tilde{F}_n} \tilde{\mathfrak{S}}_{n+1}$$

Then
- **(i)** *if* $\mathcal{S}$ *is realizing then so is* $\mathcal{T}$*, and*
- **(ii)** *if* $\mathcal{S}$ *progresses at stage* n *then so does* $\mathcal{T}$*.*

Proof sketch. i. This is straightforward to verify, and intuitively clear since a point $\tilde{x} \in |\tilde{\mathfrak{S}}_n|$ is given "more" temporal successor than its q_n-preimages that it "represents".

ii. Suppose $\mathcal{S}$ progresses at stage n, based on some eventuality $\langle x, \varphi\rangle$ at stage $j \leq n$. One readily verifies that if $j = n$, then $\mathcal{T}$ progresses at stage n. If $j < n$ then one finds a transitive temporal successor x' of x at stage $n-1$ such that $\langle x', \varphi\rangle$ is realized at stage n, so x' satisfies $\bullet\varphi$, which is in Σ by the convention stated at the start of Section 4, hence preserved by the selection construction. The claim easily follows. ◀

Repeatedly applying Construction 9.1 as in the lemma, for increasing n, we obtain the (strong) locally finite model property. Fix a primitive recursive function $\langle m, n\rangle \mapsto s_m(n)$ such that in Construction 9.1, the size of $\tilde{\mathfrak{S}}_1$ is at most $s_{\#\Sigma}(\#\mathfrak{S}_0)$, and define

$$s^*_m(n) \coloneqq (s_m)^{n+1}(1).$$

▶ **Proposition 9.4.** *A formula $\varphi \in \Sigma$ is satisfied in a DPM iff it is satisfied in a locally $s^*_{\#\Sigma}$-small quasi-sequence model.*

However, for the enumeration procedure we want to use reduced quasi-sequences, and reducing a quasi-sequence invalidates the s^* size bound, a problem we will solve in the next section.

10 Selection and Reduction

In this section we interleave the selection construction from the previous section with jump reductions, as introduced in Section 7, to produce locally $s^*_{\#\Sigma}$-small reduced quasi-sequence models.

The basic construction is simple. Start with a tree-unravelled deterministic realizing sequence. Then repeatedly apply the following. Let n be the first index in the sequence such that quasi-state n has size strictly greater than $s^*_{\#\Sigma}(n)$. Apply Construction 9.1 to the quasi-states $n-1, n, n+1$ like in Lemma 9.3. Next, apply any applicable $\langle i, j, E\rangle$-jump reductions for $i < j \leq n$.

Note that the last step can make the initial segment that was already selected shorter. However, using a measure we prove that this construction leaves initial segments of increasing lengths unmodified, so that we obtain, in the limit, a long quasi-sequence, which is, by construction, locally $s^*_{\#\Sigma}$-small and reduced.

A *working sequence* is a pair $\langle \mathcal{S}, n\rangle$ where $\mathcal{S}$ is the quasi-sequence at the start of one iteration in the construction above and n is the index as before. Formally, $\langle \mathcal{S}, n\rangle$ is a *working sequence* iff

- **(i)** $\mathcal{S} = \langle \mathfrak{S}_-, F_-\rangle$ is a quasi-sequence model,
- **(ii)** $\mathcal{S}$ is determinism-tree-like,
- **(iii)** if $n > 0$ then $F_{n-1}(D_{n-1})$ is downward closed in $\mathfrak{S}_n$,
- **(iv)** the initial segment of length n of $\mathcal{S}$ is reduced and locally $s^*_{\#\Sigma}$-small, and
- **(v)** the tail of $\mathcal{S}$ starting at index n is deterministic and tree-unravelled.

Given such a working sequence $\langle \mathcal{S}, n\rangle$, we define its *measure* to be $\langle m_1, m_2, n\rangle$ where

- $m_1 = 0$ if $\mathcal{S}$ does not progress at a stage $< n$; otherwise $m_1 = k+1$ where k is the largest index $< n$ such that $\mathcal{S}$ progresses at stage k, and
- $m_2 = 0$ if $\mathcal{S}$ does not progress at any stage $> n$; otherwise $m_2 = k - n$ where k is the least index $> n$ such that the quasi-sequence progresses at stage k.

We order measures lexicographically with standard (increasing) order in the first and third component and reversed (decreasing) order in the second component.

▶ **Construction 10.1.** Let $\langle \mathcal{S}, n\rangle$ be a working sequence as above, with $n > 0$ and measure $\langle m_1, m_2, n\rangle$. We construct a working sequence $\langle \mathcal{S}', n'\rangle$ with strictly larger measure $\langle m_1', m_2', n'\rangle$ as follows.

First, notice that the conditions of Construction 9.1 are satisfied for the tail of $\mathcal{S}$ starting at index $n-1$. Define $\tilde{\mathfrak{S}}_n$, $\tilde{F}_{n-1}$, $\tilde{F}_n$ and $\mathcal{T}$ as in Lemma 9.3. Using Lemmata 9.2 and 9.3 it follows that $\langle \mathcal{T}, n+1\rangle$ satisfies all the conditions for a working sequence, except that the initial segment of length $n+1$ need not be reduced. If it is, define $\mathcal{S}' \coloneqq \mathcal{T}$ and $n' \coloneqq n+1$. If not, there exist $i < n$ and a simulation E of $\mathfrak{S}_i$ in $\tilde{\mathfrak{S}}_n$, such that

(i) no progress occurs at stages k with $i < k \leq n$, and
(ii) there is an eventuality at a stage $\leq i$ which isn't realized at a stage $\leq i$.

Choose such i minimally. Define $\mathcal{S}'$ to be the $\langle i, n, E\rangle$-jump reduction of $\mathcal{T}$ and $n' \coloneqq i+1$. It is easy to check that $\langle \mathcal{S}', n'\rangle$ is a working sequence. ⌟

▶ **Lemma 10.2.** *In Construction 10.1, $\langle \mathcal{S}', n'\rangle$ has measure strictly larger than $\langle \mathcal{S}, n\rangle$.*

While Construction 10.1 requires $n \geq 1$, it is easily modified for $n = 0$ by considering an "imaginary" $\mathfrak{S}_{-1}$, consisting of a single point, in front of $\mathcal{S}$, and mapping that point to a root of $\mathfrak{S}_0$. We obtain the following lemma.

▶ **Lemma 10.3.** *Let $\mathcal{S}$ be a tree-unravelled deterministic realizing sequence. Then there exists a working sequence $\langle \mathcal{S}', 1\rangle$ such that every $\varphi \in \Sigma$ is satisfied in the initial quasi-state of $\mathcal{S}$ iff it is satisfied in the initial quasi-state of $\mathcal{S}'$.*

Now, iteratively applying Construction 10.1, in the limit we obtain a locally $s_{\#\Sigma}$-small and reduced quasi-sequence.

▶ **Lemma 10.4.** *Let $\varphi \in \Sigma$ be satisfied in a DPM. Then φ is satisfied in the initial quasi-state of a long reduced locally $s^*_{\#\Sigma}$-small determinism-propagating determinism-tree-like quasi-sequence $\mathcal{S}_\infty$.*

Proof sketch. By tree unravelling and Lemma 10.3 we obtain a working sequence $\langle \mathcal{S}_0, 1\rangle$ satisfying φ in the initial state. Repeatedly applying Construction 10.1 we obtain a sequence of working sequences $\langle \mathcal{S}_-, n_-\rangle$ with strictly increasing measure. It suffices to prove that for every $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that the initial segments of length k is constant from m_k onwards, which follows straightforwardly from the fact that the measures strictly increase. ◀

Since by Proposition 7.11 $\mathcal{S}_\infty$ is realizing, we obtain the following completeness result.

▶ **Proposition 10.5.** *Let $\varphi \in \Sigma$. Then φ is satisfied in a DPM iff it is satisfied in the initial quasi-state of a long reduced locally $s^*_{\#\Sigma}$-small determinism-propagating determinism-tree-like quasi-sequence $\mathcal{S}_\infty$.*

11 Enumeration

With the completeness result of Proposition 10.5, we can give an enumeration procedure for Alexandrov DTL. The key observation is that all conditions on the quasi-sequences in this completeness result are *local*, so that the long quasi-sequence exists iff short quasi-sequences with the same properties exist for all lengths.

▶ **Lemma 11.1.** *Let $\varphi \in \Sigma$. Then φ is satisfied in a DPM iff for all $n \in \mathbb{N}$, there exists a short determinism-propagating reduced locally $s^*_{\#\Sigma}$-small determinism-tree-like quasi-sequence of length n.*

Proof sketch. View these short quasi-sequences, up to isomorphism, as forming a tree-like partial order, by ordering $\mathcal{S} \sqsubseteq \mathcal{T}$ iff $\mathcal{S}$ is an initial segment of $\mathcal{T}$. By the size bound $s^*_{\#\Sigma}$, this tree is finitely branching, in that every quasi-state has only finitely many immediate successors. By Kőnig's lemma, there exists an infinite path in it, which gives the desired long quasi-sequence. ◀

Because such quasi-sequence of a certain length n has a computably bounded size, it is decidable whether such structure exists. Existence of such quasi-sequences for all finite lengths is therefore Π^0_1 (or co-computably-enumerable), hence so is satisfiability in DPMs. We conclude that validity is computably enumerable.

▶ **Theorem 11.2.** *The set of formulas of DTL valid over the class of Alexandrov spaces is computably enumerable.*

12 Concluding Remarks

We have shown that DTL of Alexandrov spaces is computably enumerable, settling an open question dating more than twenty years. By Craig's trick [8], this logic thus has a decidable axiomatization in principle, although we know that it is not finitely axiomatizable [18]. Nevertheless, there might be a natural axiomatization even if it is not finite. Here we note that a deductive calculus can be non-finite in two ways: the set of axioms could be infinite (but decidable, or at least c.e.), or derivations themselves could be infinite. An infinitary proof system in the second sense has been provided by Chopoghloo and Moniri [6], but our results open the door for an axiomatization that is infinite only in the first sense. Such axiomatizations can be quite transparent, as witnessed by cases such as the μ-calculus [39].

The axiomatization for general DTL is also infinite in the first sense [19]. Note that the latter uses an extension of the language of DTL with a polyadic "tangle" modality [9, 16]. Closely related to DTL, intuitionistic linear temporal logic may also be interpreted over topological spaces [20] and has been completely axiomatized in the Alexandrov case [21]. One peculiarity of this proof is that co-implication plays an essential role, as implication alone does not yield a complete calculus. We suspect that an axiomatization of Alexandrov DTL will involve both tangle-like operators and a "downward-looking" modality to play the role of co-implication. At this stage, this is speculative and finding a natural axiomatization remains an interesting open problem.

There are still various DTLs of interest whose axiomatizability remains open and whose logic is known to be distinct from that of all spaces or all Alexandrov spaces. DTL with finitely many iterations *is* decidable [24], but this logic does not coincide with the logic of *finite dynamical systems*, in that systems with finitely many iterations are based on a partial function. We remark that every finite topological space is Alexandrov, although the converse is not true, and thus the axiomatizability or even the decidability of DTL for finite dynamical systems remains open (although the logic with homeomorphisms is non-axiomatizable [27]). DTL of Euclidean space is different from either Alexandrov or general DTL [17] and its axiomatizability is unknown. The DTL of the real line is of particular interest, since the near-next fragment is already distinct from that of general spaces [28, 33] and its decidability is unknown, in contrast to the case of dimension greater than one, where the near-next fragment coincides with the general case [37, 13].

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